5.3 Law of Total Expectation

Agenda

- Conditional Expectation
- Law of Total Expectation (LTE)
- Law of Total Probability (Continuous version)

Conditional Expectation

Conditional Expectation: Let X be a discrete random variable. Then, the conditional expectation of X given A is

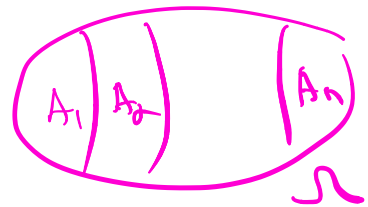
$$\mathbb{E}[X \mid A] = \sum_{x \in \Omega_X} x\mathbb{P}(X = x|A)$$

Linearity of expectation still applies to conditional expectation: $\mathbb{E}[X + Y \mid A] = \mathbb{E}[X \mid A] + \mathbb{E}[Y \mid A]$

Law of Total Expectation

Law of Total Expectation (Event Version): Let X be a random variable, and let events $A_1, \ldots, A_n$ partition the sample space. Then,

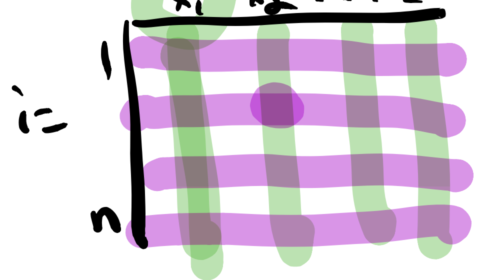
$$\mathbb{E}[X]=\sum_{i=1}^{n}\mathbb{E}[X \mid A_i]\,\mathbb{P}(A_i)$$



$$E(X)=\sum_{x\in\Omega_X} x \, \underbrace{\overbrace{\Pr(X=x)}^{p_X(x)}}_{\overset{LTP}{=} \sum_{i=1}^{n}\Pr(X=x \mid A_i)\Pr(A_i)}$$

interchange order of summation

$$= \sum_{i=1}^{n}\Pr(A_i)\underbrace{\boxed{\sum_{x\in\Omega_X} x \Pr(X=x \mid A_i)}}_{\text{by defn } E(X\mid A_i)} = \sum_{i=1}^{n}E(X\mid A_i)\Pr(A_i)$$



Linearity of expectation applies

To conditional expectation too!!

E(X+ Y | A) = E(X | A) + E(Y | A)

E(aX + b | A)= a E(X | A) + b

Law of total Expectation (RV version)

Law of Total Expectation (Event Version): Let X be a random variable, and let events $A_1, \ldots, A_n$ partition the sample space. Then,

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid A_i]\, \mathbb{P}(A_i)$$

Law of Total Expectation (RV Version): Suppose X and Y be discrete random variables. Then,

$$\mathbb{E}[X] = \sum_{y \in \Omega_Y} \mathbb{E}[X \mid Y = y]\, p_Y(y)$$

$\Omega_Y = \{y_1, y_2, \ldots$

$Y = y_1$

$Y = y_2$

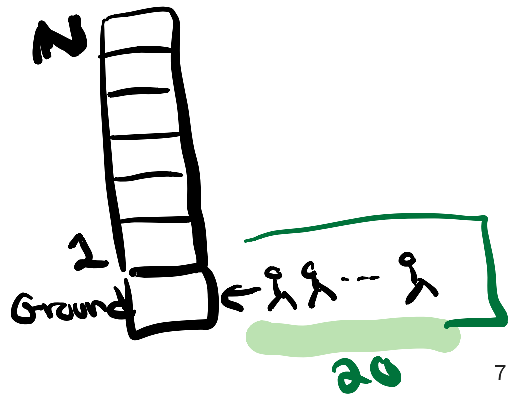
$\vdots$

Problem

The number of people who enter an elevator on the ground floor is a Poisson random variable with mean 10. If there are N floors above the ground floor, and if each person is equally likely to get off at any one of the N floors, independently of where the others get off, compute the expected number of stops that the elevator will make before discharging all the passengers.

Y # stops elevator make.

$E(Y) = ?$



X : # people who enter elevator

$X \sim Poi(10)$

Special case problems

Suppose 20 people enter elevator.

$$Y_i = \begin{cases} 1 & \text{elevator stops on } i^{th} \text{ floor} \\ 0 & \text{o.w.} \end{cases}$$

$E(Y_i) = \Pr(\text{elevator stops at } i^{th} \text{ floor})$

$[i \in \{1, \dots, N\}]$

$$Y_i = \begin{cases} 1 & \text{elevator stops on } i^{th} \text{ floor} \\ 0 & \text{otherwise} \end{cases}$$

$E(Y_i) = ?$

a) $\frac{20}{N}$

b) $\boxed{1 - \left(1 - \frac{1}{N}\right)^{20}}$

c) 1

d) I don't know

$= \Pr(\text{somebody chooses } i^{th} \text{ floor})$

≥ 1 person

$= 1 - \Pr(\text{nobody chooses that floor})$

$= 1 - \left(1 - \frac{1}{N}\right)^{20}$

who stop on i^{th} floor

$\sim \text{Bin}\left(20, \frac{1}{N}\right)$

20 people

N floors.

k people $\Rightarrow$ $\boxed{E(Y_i) = \Pr(\text{elev stops at } i^{th} \text{ floor}) = 1 - \left(1 - \frac{1}{N}\right)^k}$

X # people enter $\sim Poi(10)$

Y # stops elev.

$$E(Y) = \sum_{k=0}^{\infty} \underbrace{E(Y \mid X=k)} \underbrace{Pr(X=k)}_{e^{-10}\frac{10^k}{k!}}$$

LTE

simplifying r.v. Y

$$E(Y_1 + Y_2 + \dots + Y_N \mid X=k)$$

$$Y_i = \begin{cases} 1 & \text{elev stops at i-th floor} \\ 0 & \text{o.w.} \end{cases}$$

$$\overset{LoE}{=} E(Y_1 \mid X=k) + E(Y_2 \mid X=k) + \dots + E(Y_N \mid X=k)$$

$$= N\left(1 - \left(1 - \frac{1}{N}\right)^k\right)$$

$$= \sum_{k=0}^{\infty} N\left(1 - \left(1 - \frac{1}{N}\right)^k\right) e^{-10} \frac{10^k}{k!}$$

Solution

X number of people who enter

Y number of stops

$$Pr(X = k) = e^{-10}\frac{10^k}{k!}$$

$$E(Y) = \sum_{k=0}^{\infty} E(Y|X = k)P(X = k)$$

$$E(Y|X = k) = E(Y_1 + \ldots + Y_N|X = k)$$

Y_i indicates a stop on floor i

$$E(Y_i|X = k) = \left(1 - (1 - 1/N)^k\right)$$

LTP discrete (r.v.)

$$\boxed{Pr(E) = \sum_{y \in \mathcal{R}_Y} Pr(E \mid Y=y) \underbrace{Pr(Y=y)}_{p_Y(y)}}$$

LAW OF TOTAL PROBABILITY (CONT VERSION)

$$Pr(E) = \int_{-\infty}^{\infty} Pr(E \mid Y=y) \underbrace{f_Y(y)\, dy}_{\approx Pr(y \le Y \le y+dy)} \qquad (*)$$

$X_1, X_2, \ldots, X_k$ i.i.d. density f CDF F

$$\boxed{Pr(X_1 = \min(X_1, \ldots, X_k))}$$

$$= Pr(\boxed{X_1 < X_2, X_1 < X_3, \ldots, X_1 < X_k})$$

LTP

$$= \int_{-\infty}^{\infty} Pr(X_2 > X_1, X_3 > X_1, \dots, X_k > X_1 \mid X_1 = x) f(x)\, dx$$

$$= Pr(X_2 > x, X_3 > x, \dots, X_k > x \mid X_1 = x)$$

$$= \prod_{i=2}^{k} \underbrace{Pr(X_i > x)}_{1 - \underbrace{Pr(X_i \le x)}_{F(x)}}$$

$$\prod_{i=2}^{k} Pr(X_i > x) = ?$$

$\Rightarrow$ a) $(1-F(x))^{k-1}$
b) $F(x)^{k-1}$
c) $(1-F(x))^{k}$
d) I don't know

$$= (1-F(x))^{k-1}$$

$$= \int_{-\infty}^{\infty} \underbrace{(1-F(x))^{k-1}}_{u^{k-1}} \underbrace{f(x)dx}_{-du}$$

$du = ?$

a) $f(x)$
b) $f(x)dx$
c) $\boxed{-f(x)dx}$
d) I don't know

change of variable

$u = 1 - F(x)$

$du = -f(x)\,dx$

as x goes $-\infty \to \infty$

u goes from $1 \to 0$

$$\frac{d}{dx} F(x) = f(x)$$

$$F(x) = \int_{-\infty}^{x} f(w)\,dw$$

$F(x) \to \underset{-\infty}{0} \to \underset{\infty}{1}$

u goes from

a) $-\infty \to \infty$

b) $0 \to 1$

c) $1 \to 0$

d) I don't know

$$-\int_1^0 u^{k-1}\,du = \int_0^1 u^{k-1}\,du = \frac{u^k}{k}\Big|_0^1$$

$$= \frac{1}{k}$$

$$\boxed{\Pr(X_1 = \min(X_1, \dots, X_k)) = \frac{1}{k}}$$

$$\Pr(X_k = \min(X_1 \dots X_k)) = \frac{1}{k}$$

Multivariate: From Discrete to Continuous

	Discrete	Continuous
Joint PMF/PDF	$p_{X,Y}(x,y) = P(X = x, Y = y)$	$f_{X,Y}(x,y) \neq P(X = x, Y = y)$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t \le x} \sum_{s \le y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s) ds dt$
Normalization	$\sum_{x} \sum_{y} p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx dy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_{y} p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$
Expectation	$E[g(X,Y)] = \sum_{x} \sum_{y} g(x,y) p_{X,Y}(x,y)$	$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$
Conditional PMF/PDF	$p_{X\mid Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$	$f_{X\mid Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
Conditional Expectation	$E[X \mid Y = y] = \sum_{x} x p_{X\mid Y}(x \mid y)$	$E[X \mid Y = y] = \int_{-\infty}^{\infty} x f_{X\mid Y}(x \mid y) dx$
Independence	$\forall x, y, p_{X,Y}(x,y) = p_X(x) p_Y(y)$	$\forall x, y, f_{X,Y}(x,y) = f_X(x) f_Y(y)$

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Law of Total Expectation (Example from last time)

Show that if $X \sim Geo(p)$, then $\mu = E[X] = 1/p$ by using the LTE conditioning on the first flip.

$$\mu = E[X] = E[X \mid H]P(H) + E[X \mid T]P(T) \qquad \text{(LTE)}$$

Law of Total Expectation (Example from last time)

Show that if $X \sim Geo(p)$, then $\mu = E[X] = 1/p$ by using the LTE conditioning on the first flip.

$$\mu = E[X] = E[X \mid H]P(H) + E[X \mid T]P(T) \qquad \text{(LTE)}$$

$$= 1 \cdot p + (E[1 + X]) \cdot (1 - p)$$

$$= p + (1 + E[X]) \cdot (1 - p)$$

So,

$$\cancel{\mu} = p + (1 + \mu)(1 - p) = p + 1 - p + \mu - \mu p = 1 + \cancel{\mu} - \mu p$$

$$0 = 1 - \mu p \rightarrow \mu = \frac{1}{p}$$