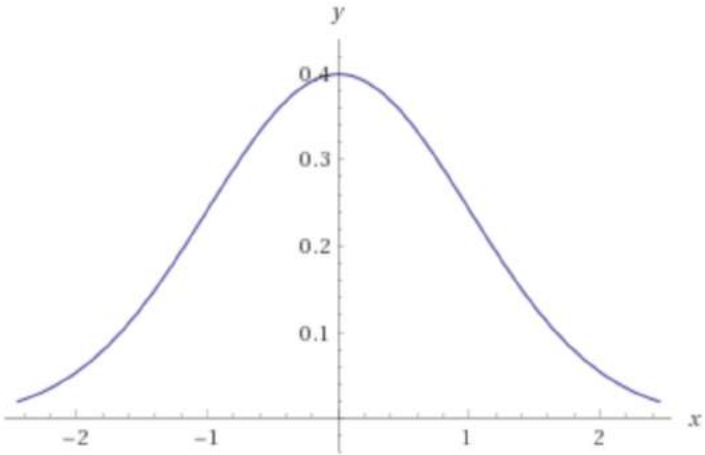
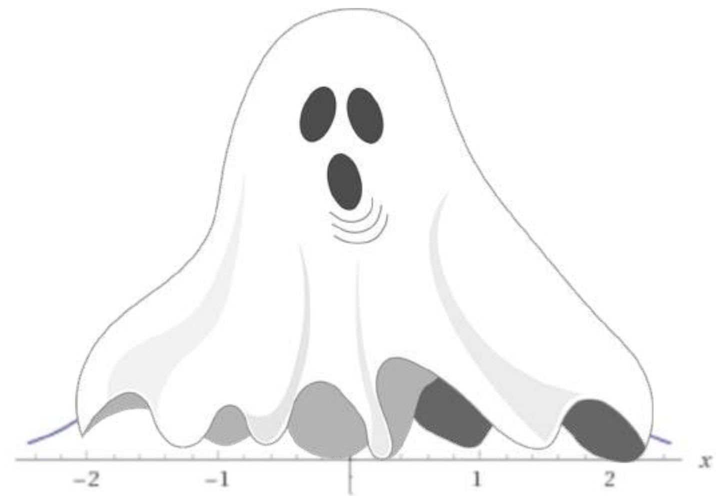


Normal Distribution



Paranormal Distribution

Anna Karlin
Most Slides by Alex Tsun

Agenda

- The Normal/Gaussian RV
- Closure properties of the Normal RV
- The standard normal CDF
- The Central Limit Theorem!

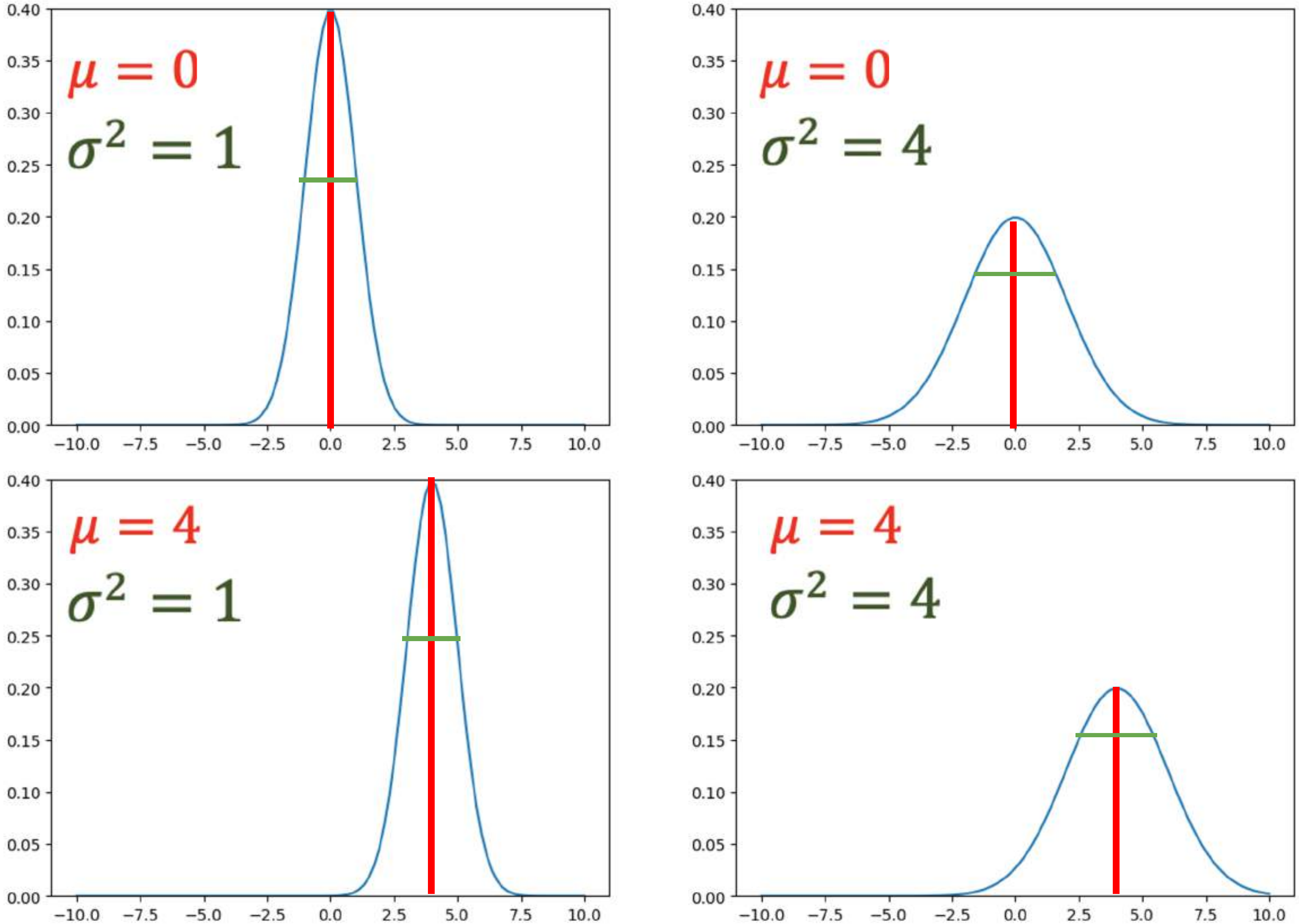
The Normal/Gaussian RV

Normal (Gaussian, "bell curve") Distribution: $X \sim \mathcal{N}(\mu, \sigma^2)$ if and only if X has the following pdf:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

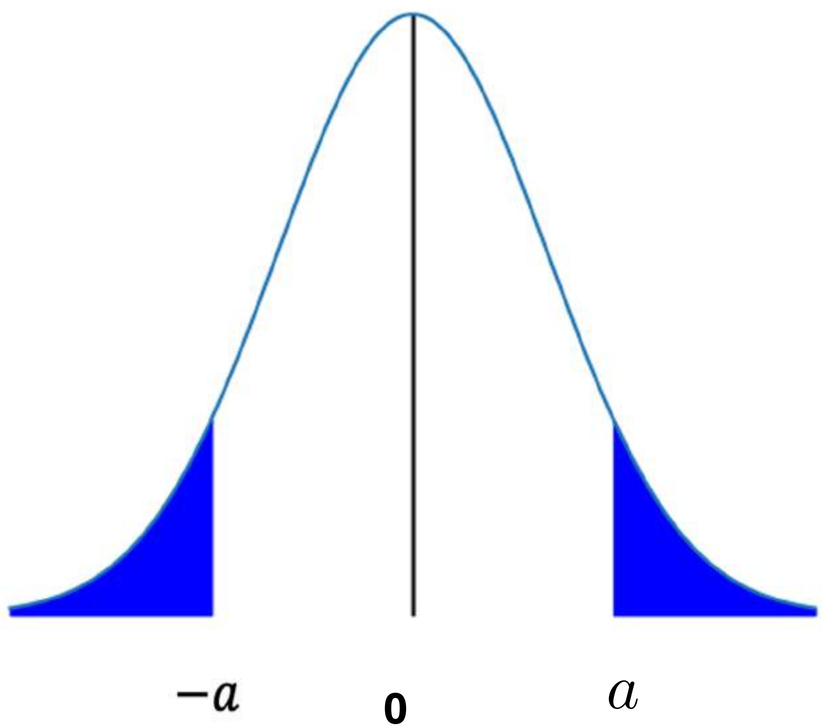
$$E[X] = \mu \qquad\qquad Var(X) = \sigma^2$$

THE NORMAL PDF



The Standard Normal CDF

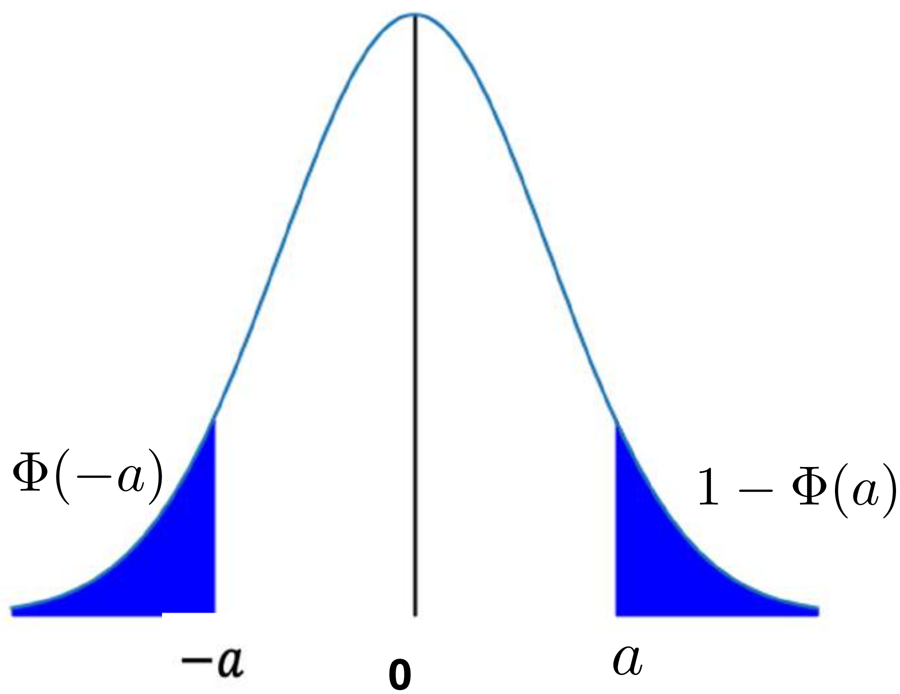
$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



The Standard Normal CDF

If $Z \sim \mathcal{N}(0,1)$, we denote the CDF $\Phi(a) = F_Z(a) = P(Z \leq a)$, since it's so commonly used. There is no closed-form formula, so this CDF is stored in a Φ table.

$$\Phi(-a) = 1 - \Phi(a)$$



The Standard Normal CDF

$$P(Z \leq 1.09) = \Phi(1.09) \approx 0.8621$$

Φ Table: $\mathbb{P}(Z \leq z)$ when $Z \sim \mathcal{N}(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

What about non-Standard Normals?

$X \sim \mathcal{N}(\mu, \sigma^2)$,

We can Standardize any RV

Let X be **ANY** random variable (discrete or continuous) with $E[X] = \mu$ and $Var(X) = \sigma^2$, and $a, b \in \mathbb{R}$. Then,

$$E[aX + b] = aE[X] + b = a\mu + b$$

$$Var(aX + b) = a^2Var(X) = a^2\sigma^2$$

In particular, we call $\frac{X-\mu}{\sigma}$ a standardized version of X, as it measures how many standard deviations above the mean a point is.

$$E\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma}(E[X] - \mu) = 0$$

$$Var\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2}Var(X - \mu) = \frac{1}{\sigma^2}\sigma^2 = 1$$

Normals stay normal! (Under scale+Shift)

Closure of the normal (Under scale+Shift)

Let X be **ANY** random variable (discrete or continuous) with $E[X] = \mu$ and $Var(X) = \sigma^2$, and $a, b \in \mathbb{R}$. Recall,

$$E[aX + b] = aE[X] + b = a\mu + b$$

$$Var(aX + b) = a^2 Var(X) = a^2\sigma^2$$

But if $X \sim \mathcal{N}(\mu, \sigma^2)$ (a Normal rv), then

$$aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$

In particular,

$$\frac{X - \mu}{\sigma} \sim \mathcal{N}(0,1)$$

Note the "special" thing here is that the transformed RV remains a Normal rv – the mean and variance are no surprise.

X is normal with mean 3 and variance 9.
What is

- $\Pr(2 < X < 5)$
- $\Pr(X > 0)$
- $\Pr(|X-3| > 6)$

X is normal with mean 3 and variance 9.
What is

- Pr (2 < X < 5)

- Pr (X > 0)

- Pr (|X-3| > 6)

Φ Table: $\mathbb{P}(Z \leq z)$ when $Z \sim \mathcal{N}(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
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2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

From $N(\mu, \sigma^2)$ to standard normal

For a $X \sim \mathcal{N}(\mu, \sigma^2)$, we have

$$F_X(y) = P(X \leq y) = P\left(\frac{X - \mu}{\sigma} \leq \frac{y - \mu}{\sigma}\right) = P\left(Z \leq \frac{y - \mu}{\sigma}\right) = \Phi\left(\frac{y - \mu}{\sigma}\right)$$

$$P(a \leq X \leq b) = F_X(b) - F_X(a) = \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right)$$

Summary: The Normal/Gaussian RV

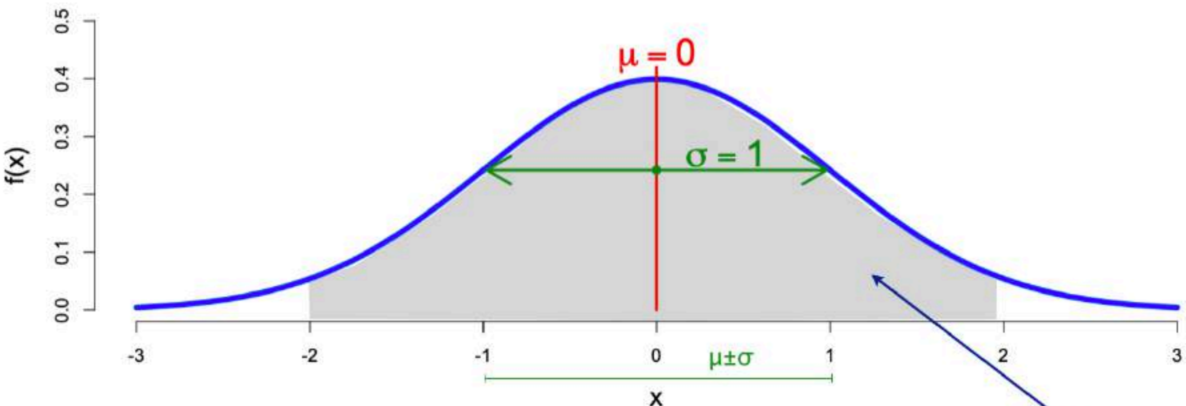
<u>**Normal (Gaussian, "bell curve") Distribution:**</u> $X \sim \mathcal{N}(\mu, \sigma^2)$ if and only if X has the following pdf:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$E[X] = \mu \qquad\qquad Var(X) = \sigma^2$$

The "standard normal" random variable is typically denoted Z and has mean 0 and variance 1. By the closure property of normals, if $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$. The CDF has no closed form, but we denote the CDF of the standard normal by $\Phi(a) = F_Z(a) = P(Z \leq a)$. Note that by symmetry of the density about 0, $\Phi(-a) = 1 - \Phi(a)$.

Normal random variables



If $Z \sim N(\mu,\sigma^2)$ what is $P(\mu-\sigma < Z < \mu+\sigma)$?

$P(\mu - \sigma < Z < \mu + \sigma) = \Phi(1) - \Phi(-1) \approx 68\%$

$P(\mu - 2\sigma < Z < \mu + 2\sigma) = \Phi(2) - \Phi(-2) \approx 95\%$

$P(\mu - 3\sigma < Z < \mu + 3\sigma) = \Phi(3) - \Phi(-3) \approx 99\%$

Why?

$N(\mu,\sigma^2)$ $N(0,1)$

$\mu - k\sigma < \boxed{Z} < \mu + k\sigma \iff -k < \boxed{(Z-\mu)/\sigma} < +k$

Closure of the normal (under addition)

Closure of the normal (under addition)

Let X, Y be **ANY independent** random variables (discrete or continuous) with $E[X] = \mu_X$, $E[Y] = \mu_Y$, $Var(X) = \sigma_X^2$, $Var(Y) = \sigma_Y^2$, and $a, b, c \in \mathbb{R}$. Recall,

$$E[aX + bY + c] = aE[X] + bE[Y] + c = a\mu_X + b\mu_Y + c$$

$$Var(aX + bY + c) = a^2 Var(X) + b^2 Var(Y) = a^2\sigma_X^2 + b^2\sigma_Y^2$$

But if $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ (both independent Normal rvs), then

$$aX + bY + c \sim \mathcal{N}(a\mu_X + b\mu_Y + c, a^2\sigma_X^2 + b^2\sigma_Y^2)$$

Note the "special" thing here is that the sum remains a Normal rv – the mean and variance are no surprise.

5.7 The Central Limit Theorem

The Sample Mean

Sample Mean: Let $X_1, X_2, \ldots X_n$ be a sequence of iid random variables with mean μ and variance σ^2. The sample mean is

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

The Sample Mean

<u>Sample Mean:</u> Let $X_1, X_2, \ldots X_n$ be a sequence of iid random variables with mean μ and variance σ^2. The sample mean is

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$E[\bar{X}_n] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n} n\mu = \mu$$

$$Var(\bar{X}_n) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$$

The Central Limit Theorem

Consider i.i.d. (independent, identically distributed) random vars $X_1, X_2, X_3, \ldots$

Where X_i has $\mu = E[X_i]$ and $\sigma^2 = Var[X_i]$

Consider random variables

$$X_1 + X_2 + \ldots + X_n$$

and

$$\frac{1}{n} \sum_{i=1}^{n} X_i$$

The Central Limit Theorem

Consider i.i.d. (independent, identically distributed) random vars
X_1, X_2, X_3, …
Where X_i has $\mu = E[X_i]$ and $\sigma^2 = Var[X_i]$

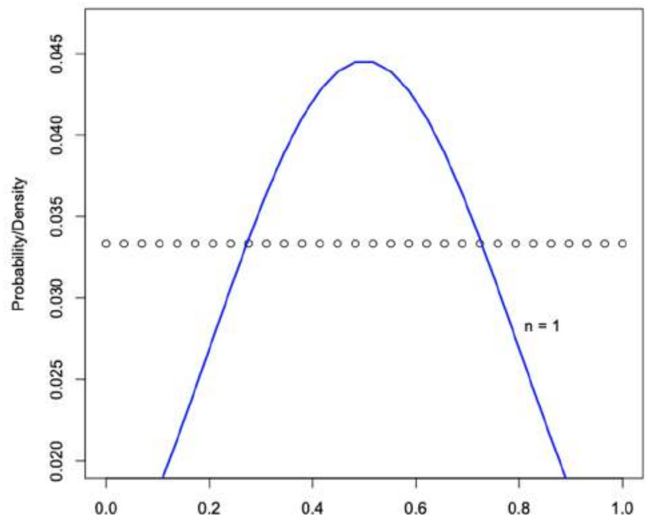
As n → ∞,

$$\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \longrightarrow N(0,1)$$

Restated: As n → ∞,

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to N\left(\mu, \frac{\sigma^2}{n}\right)$$

CLT (Pictures)



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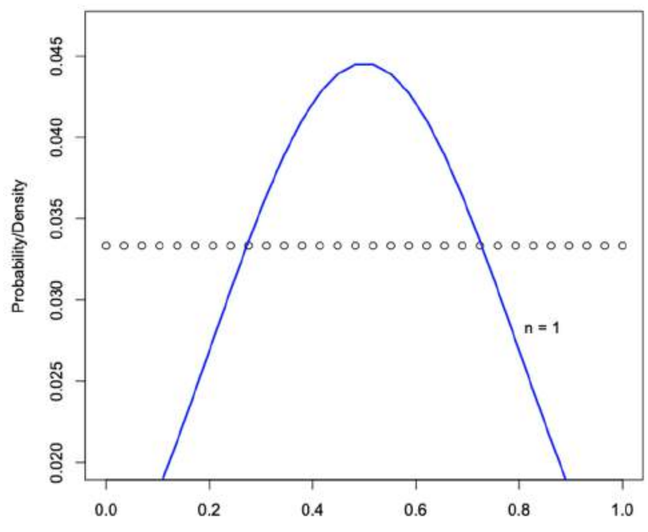

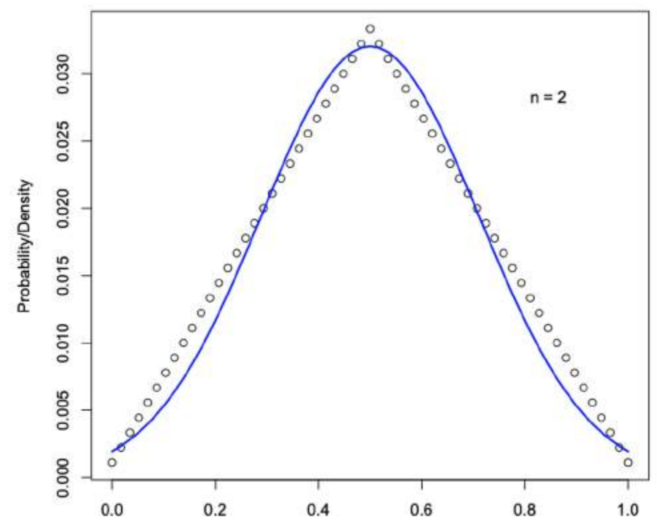

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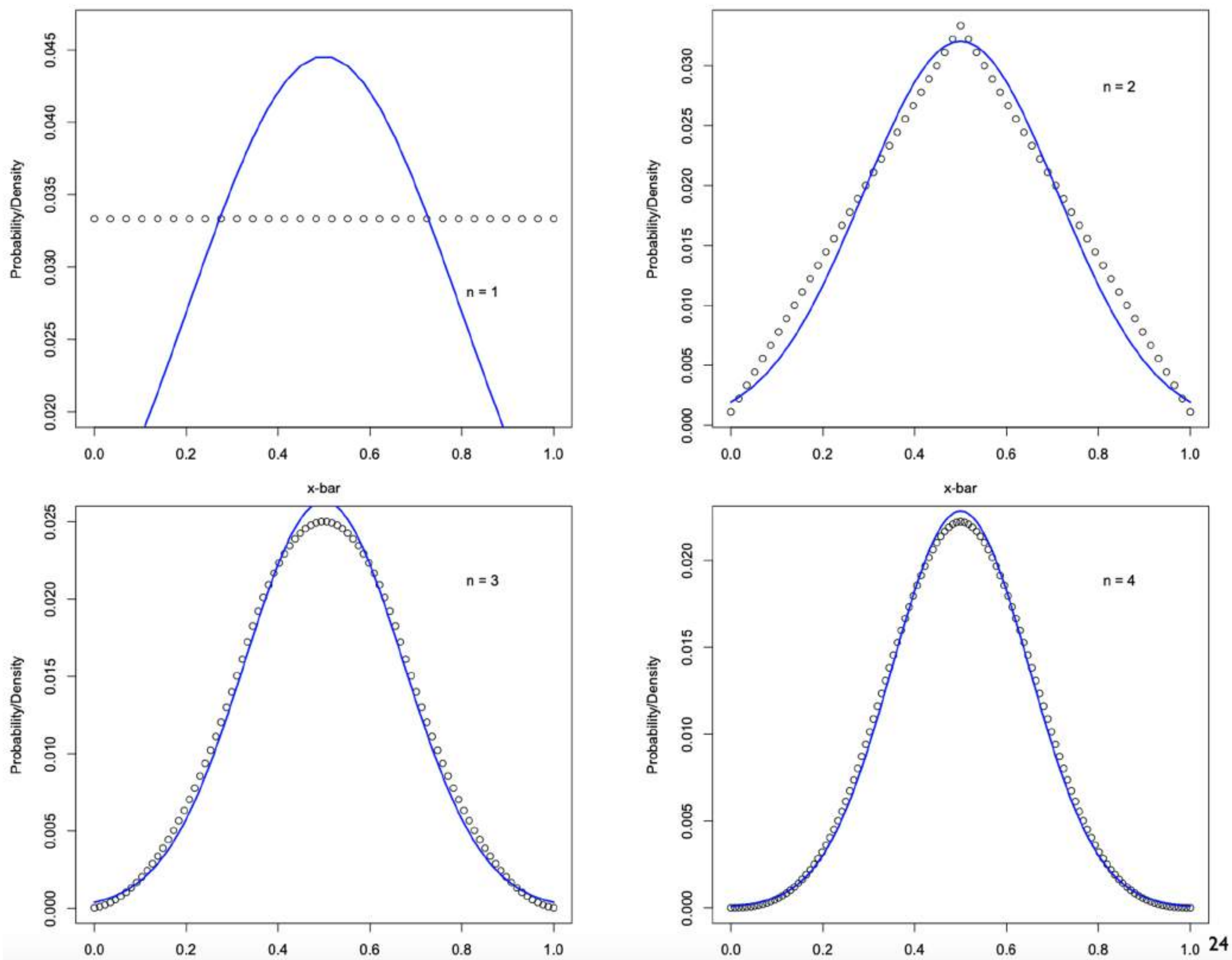
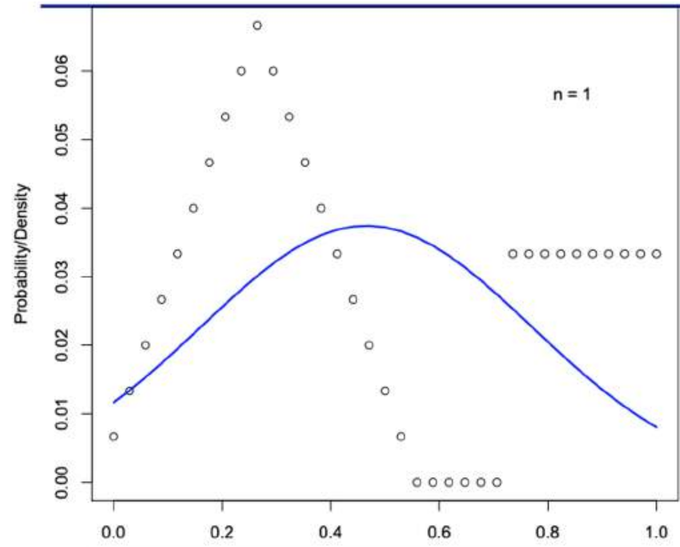


From: https://courses.cs.washington.edu/courses/cse312/17wi/slides/10limits.pdf

CLT (Pictures)



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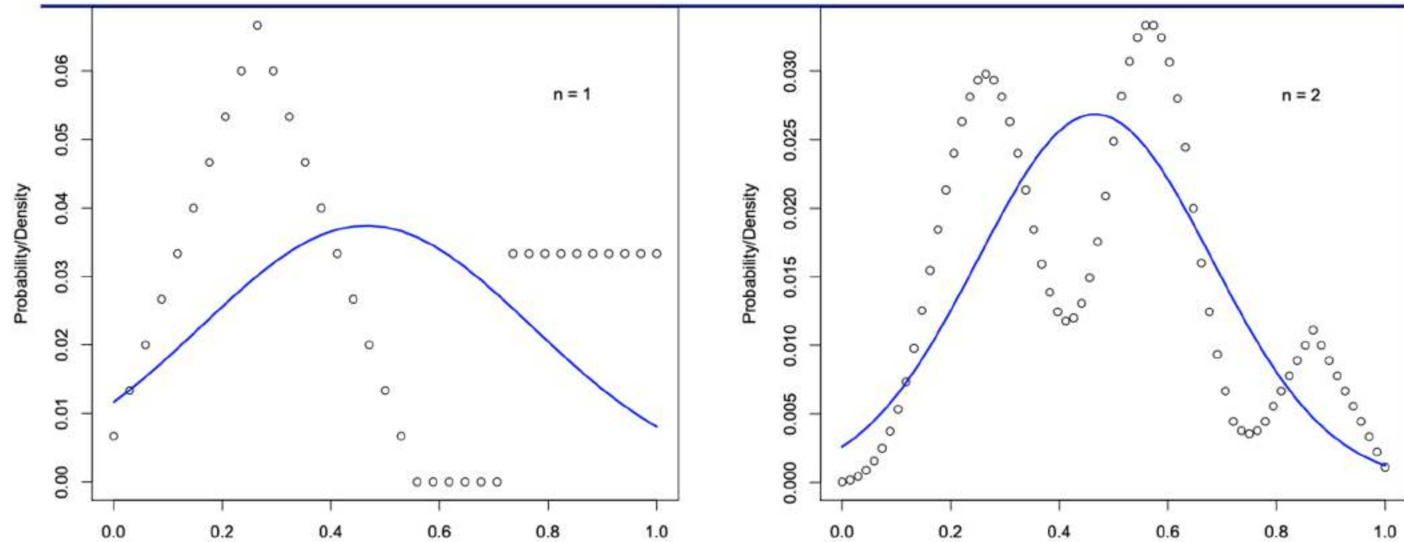

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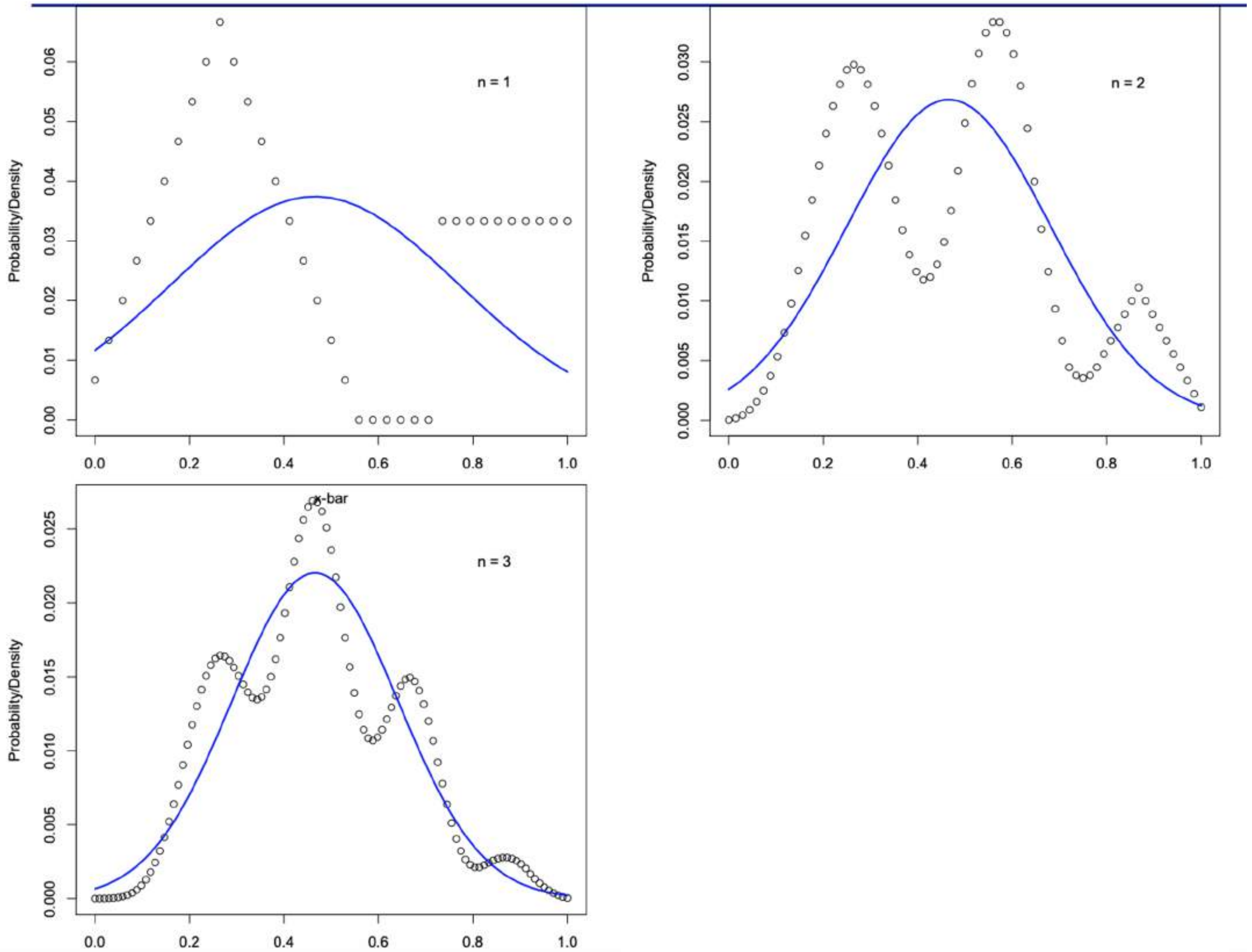

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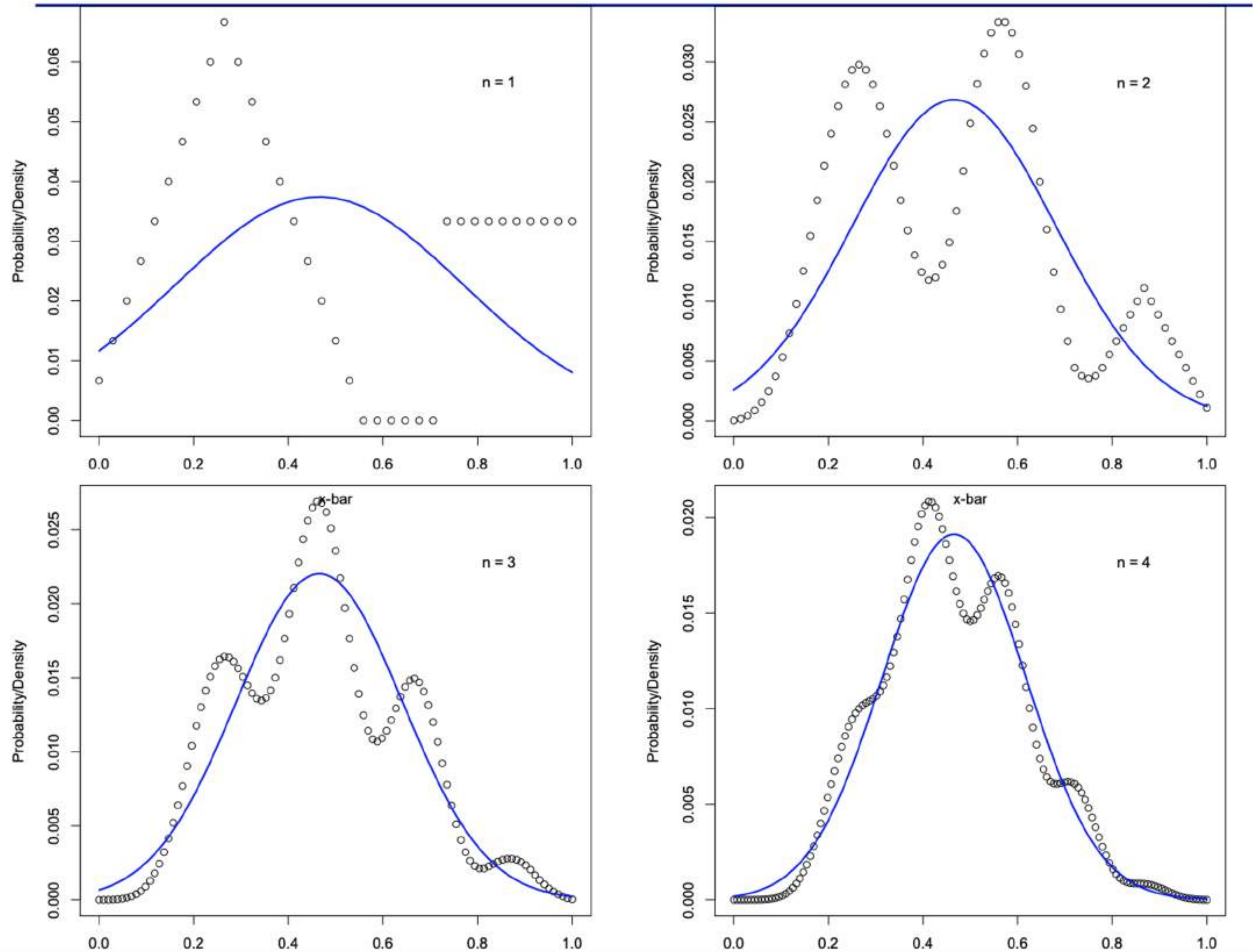

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CLT in the real world

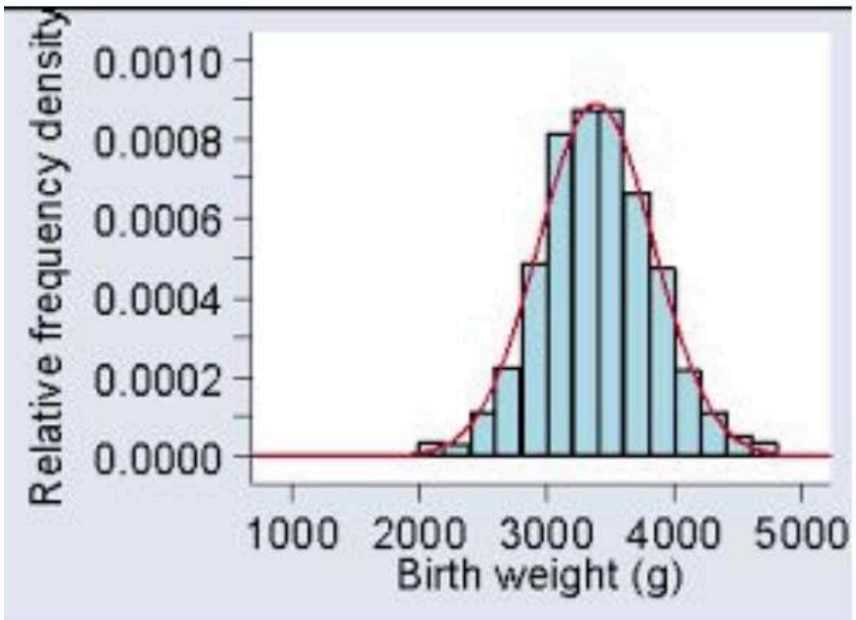
CLT is the reason many things appear normally distributed
Many quantities = sums of (roughly) independent random vars

Exam scores: sums of individual problems

People's heights: sum of many genetic & environmental factors

Measurements: sums of various small instrument errors

CLT in the real world



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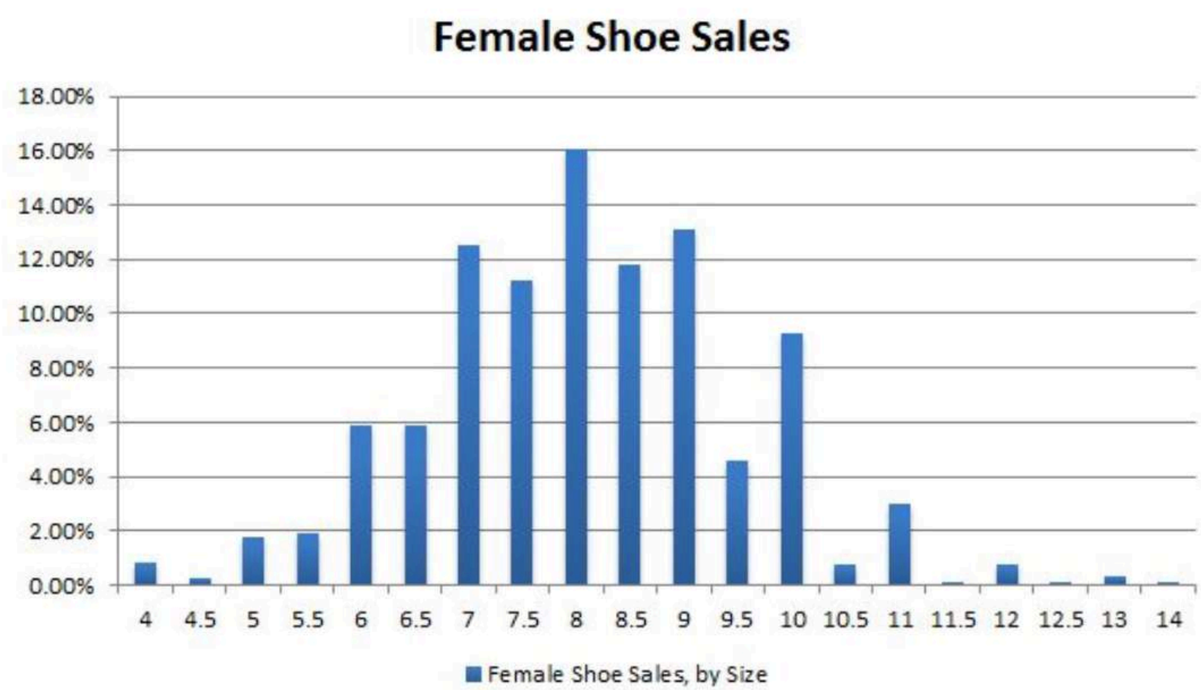

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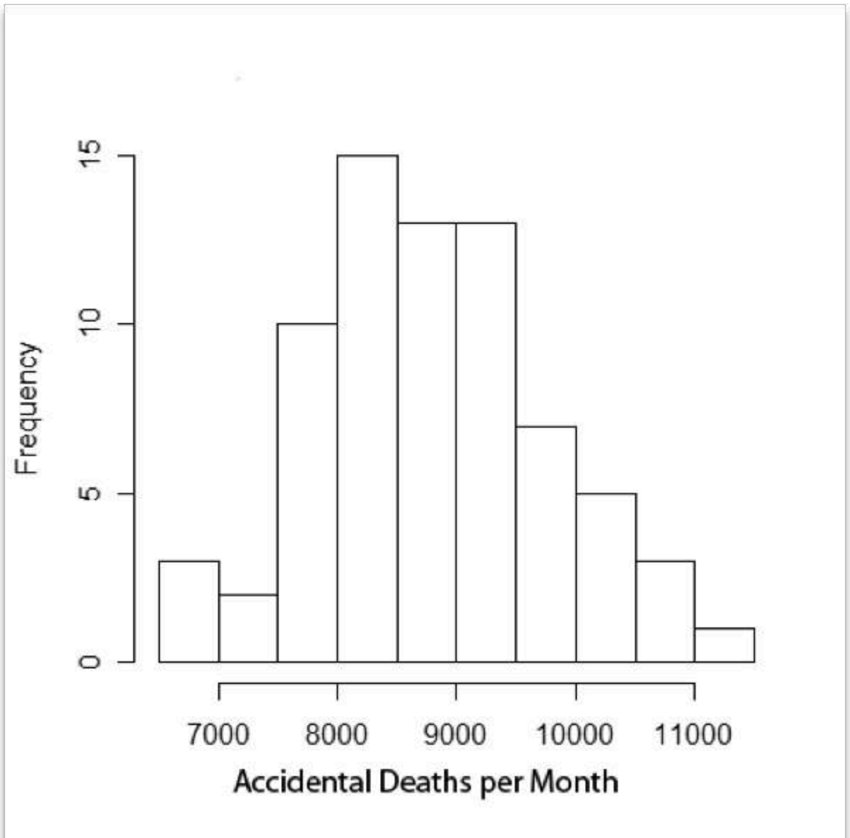
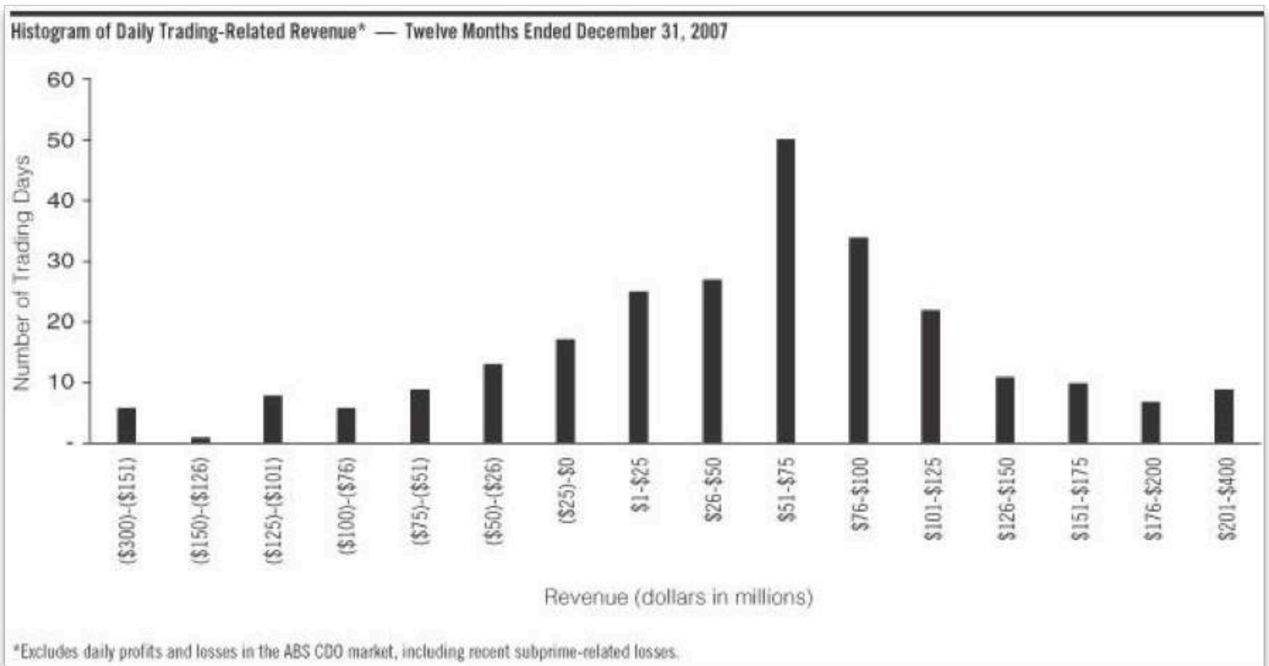


CLT in the real world



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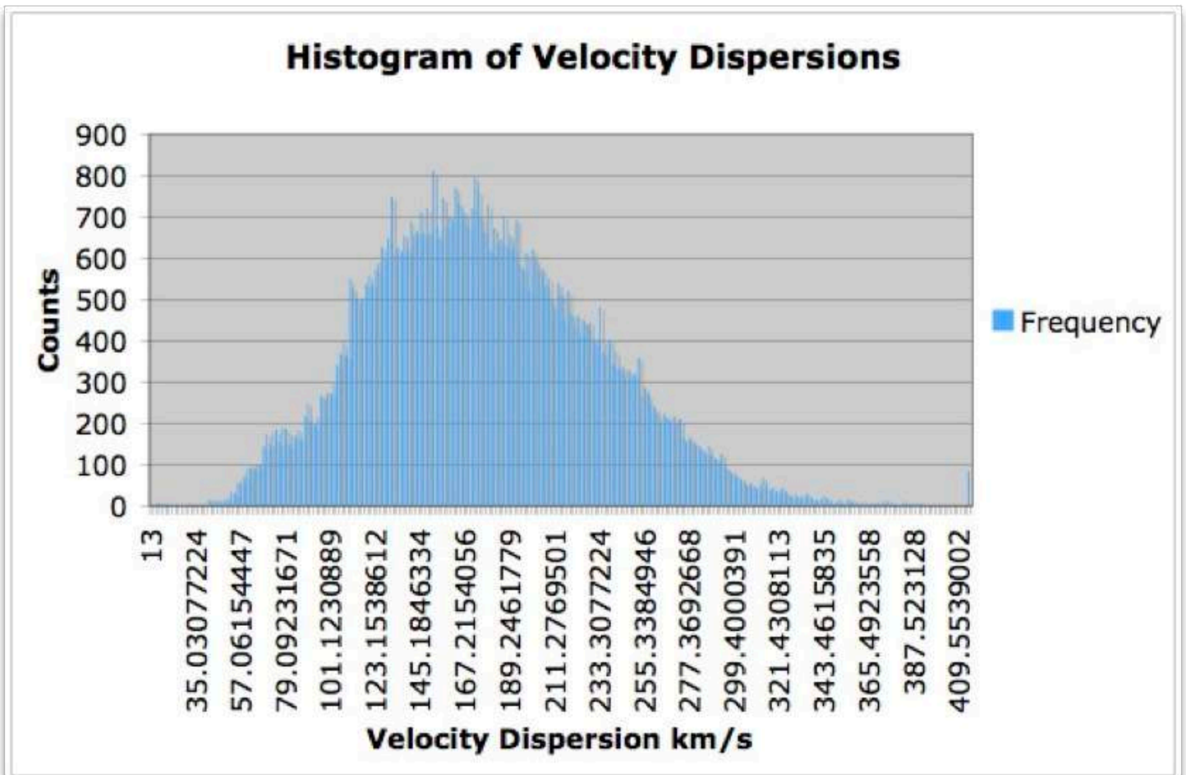

CLT (Example)

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Since X is the sum of 40 iid $Ber\left(\frac{1}{2}\right)$ rvs, we can apply the CLT. We have $E[X] = np = 40\left(\frac{1}{2}\right) = 20$ and $Var(X) = np(1-p) = 40\left(\frac{1}{2}\right)\left(1 - \frac{1}{2}\right) = 10$. So, we can use $X \approx \mathcal{N}(20, 10)$.

$$P(15 \leq X \leq 25) \approx P(15 \leq \mathcal{N}(20,10) \leq 25) = P\left(\frac{15-20}{\sqrt{10}} \leq Z \leq \frac{25-20}{\sqrt{10}}\right)$$
$$\approx P(-1.58 \leq Z \leq 1.58) = \Phi(1.58) - \Phi(-1.58) \approx 0.8862$$

The Continuity Correction (Idea)

- Suppose I asked you to estimate Pr (X = 20) using the normal approximation.
- Problem: Binomial is discrete, Normal is continuous.

The Continuity Correction (Idea)

Notice that in computing $P(15 \leq X \leq 25)$, we sum over $25 - 15 + 1 = 11$ terms of the PMF.

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Notice that in computing $P(15 \le X \le 25)$, we sum over $25 - 15 + 1 = 11$ terms of the PMF. However, our integral $P(15 \le \mathcal{N}(20,10) \le 25)$ has width 10. We'll always be off-by-one since the number of integers in $[a, b]$ is $b - a + 1$ (for integers $a \le b$).

The continuity correction says we should add 0.5 in each direction:

$$P(15 \le X \le 25) \approx P(14.5 \le \mathcal{N}(20,10) \le 25.5) = P\left(\frac{14.5 - 20}{\sqrt{10}} \le Z \le \frac{25.5 - 20}{\sqrt{10}}\right)$$
$$\approx P(-1.74 \le Z \le 1.74) = \Phi(1.74) - \Phi(-1.74) \approx 0.9182$$

(The exact answer from the previous slide was 0.9193)

The Continuity Correction

Continuity Correction (CC): When approximating an integer-valued (***discrete***) random variable X with a continuous one Y (such as in the CLT), if asked to find $P(a \leq X \leq b)$ for integers $a \leq b$, you should use $P(a - 0.5 \leq Y \leq b + 0.5)$ so that the width of the interval being integrated is the same as the number of terms summed over $(b - a + 1)$.

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Note: If you were applying the CLT to sums of continuous RVs instead, you would ***not*** apply it.

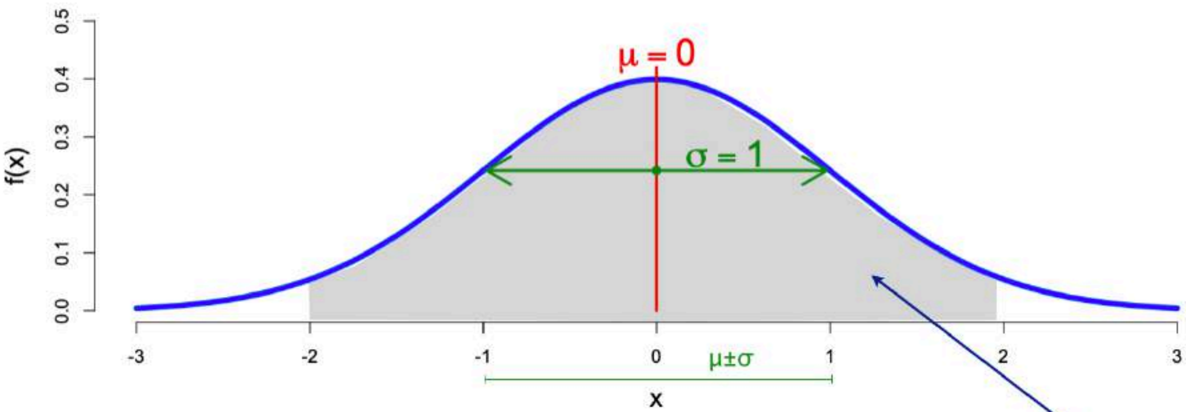
The Central Limit Theorem

Central Limit Theorem (CLT): Let $X_1, \dots, X_n$ be a sequence of iid random variables with mean μ and (finite) variance σ^2. We've seen that the sample mean $\bar{X}_n$ has mean μ and variance σ^2/n. Then, as $n \to \infty$, the following equivalent statements hold:

1. $\bar{X}_n \to \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$.
2. $\frac{\bar{X}_n - \mu}{\sqrt{\sigma^2/n}} \to \mathcal{N}(0,1)$.
3. $\sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2)$. (Not "technically" correct, but useful for applications).
4. $\frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n\sigma^2}} \to \mathcal{N}(0,1)$.

The mean or variance are no surprise; the importance of the CLT is, regardless of the distribution of the X_i's, the sample mean approaches a Normal distribution as $n \to \infty$.

Normal random variables



If $Z \sim N(\mu,\sigma^2)$ what is $P(\mu-\sigma < Z < \mu+\sigma)$?

$P(\mu - \sigma < Z < \mu + \sigma) = \Phi(1) - \Phi(-1) \approx 68\%$

$P(\mu - 2\sigma < Z < \mu + 2\sigma) = \Phi(2) - \Phi(-2) \approx 95\%$

$P(\mu - 3\sigma < Z < \mu + 3\sigma) = \Phi(3) - \Phi(-3) \approx 99\%$

Why?

$N(\mu,\sigma^2)$ ... $N(0,1)$

$\mu - k\sigma < \boxed{Z} < \mu + k\sigma \quad \Leftrightarrow \quad -k < \boxed{(Z-\mu)/\sigma} < +k$

The Standard Normal CDF

Φ Table: $\mathbb{P}(Z \leq z)$ when $Z \sim \mathcal{N}(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999