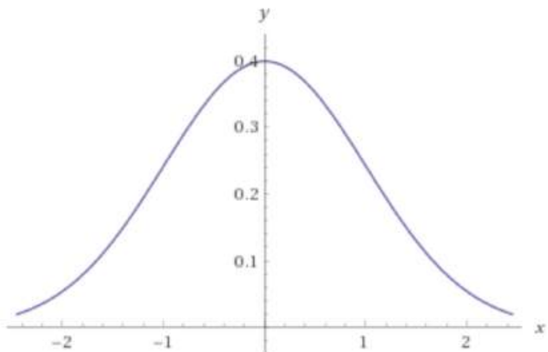
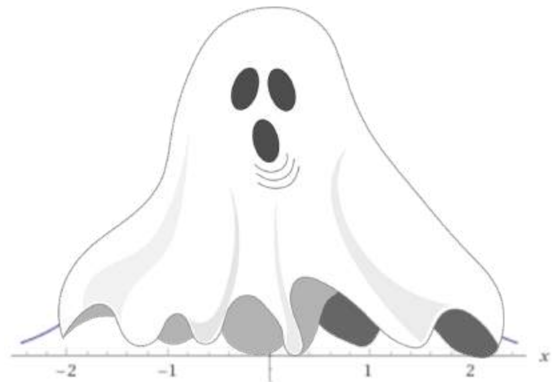


Normal Distribution



Paranormal Distribution

Anna Karlin
Most Slides by Alex Tsun

Agenda

- The Normal/Gaussian RV
- Closure properties of the Normal RV
- The standard normal CDF
- The Central Limit Theorem!

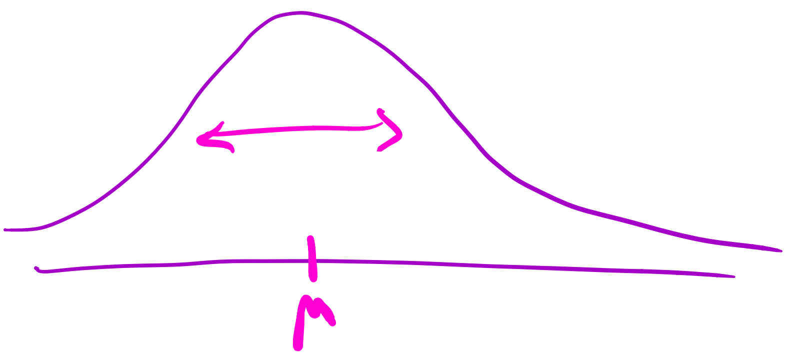
The Normal/Gaussian RV

Normal (Gaussian, "bell curve") Distribution: $X \sim \mathcal{N}(\mu, \sigma^2)$ if and only if X has the following pdf:

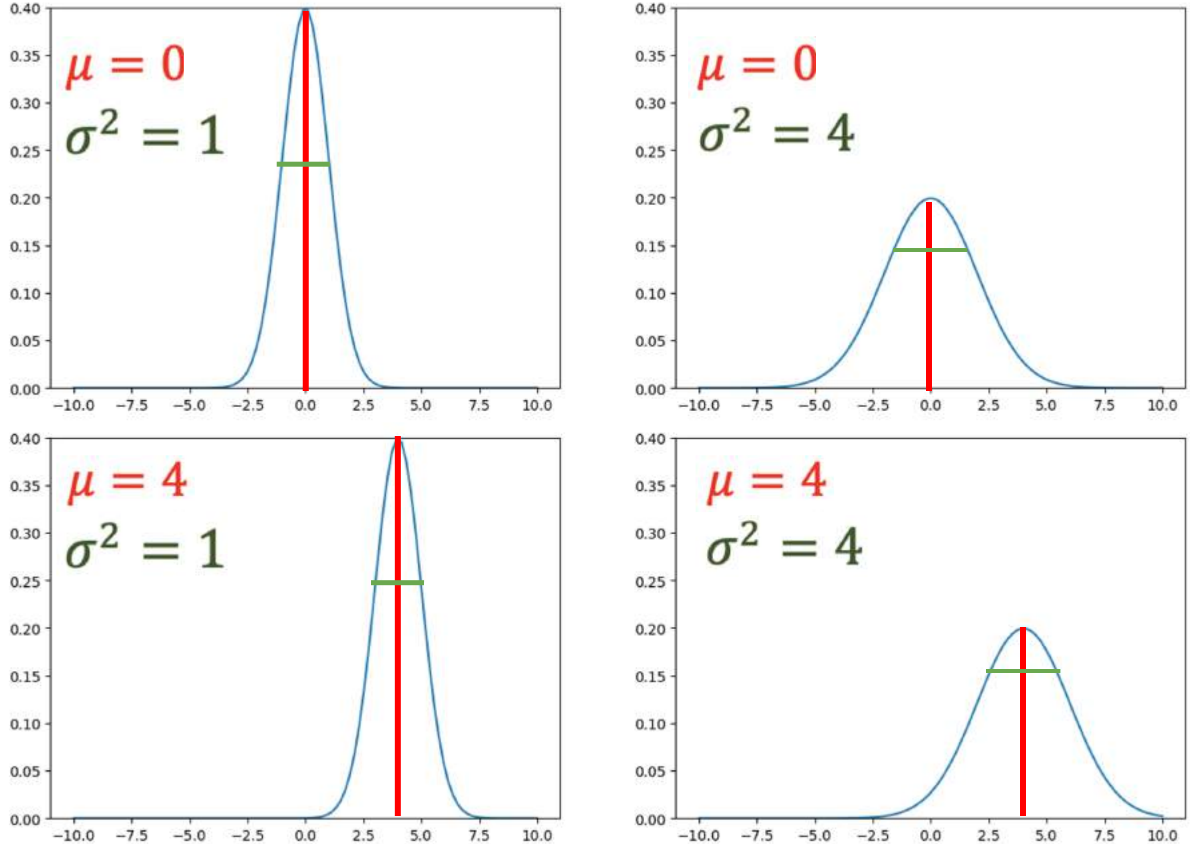
$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$x \in \mathbb{R}$

$$E[X] = \mu \qquad Var(X) = \sigma^2$$



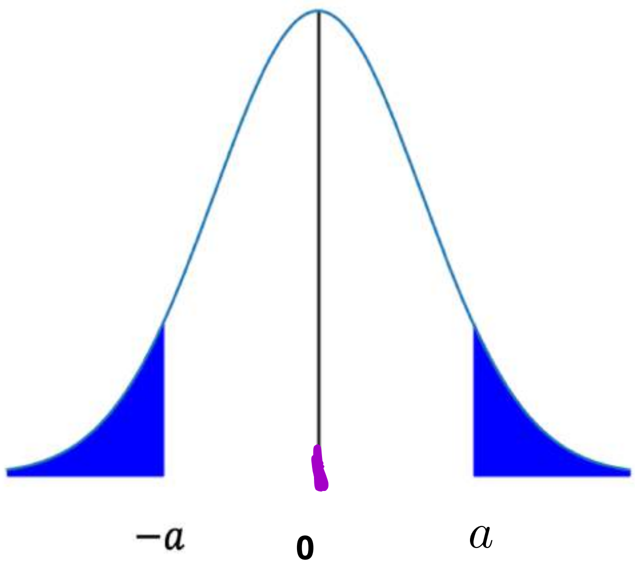
The Normal PDF



The Standard Normal CDF

$Z \sim N(0,1)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

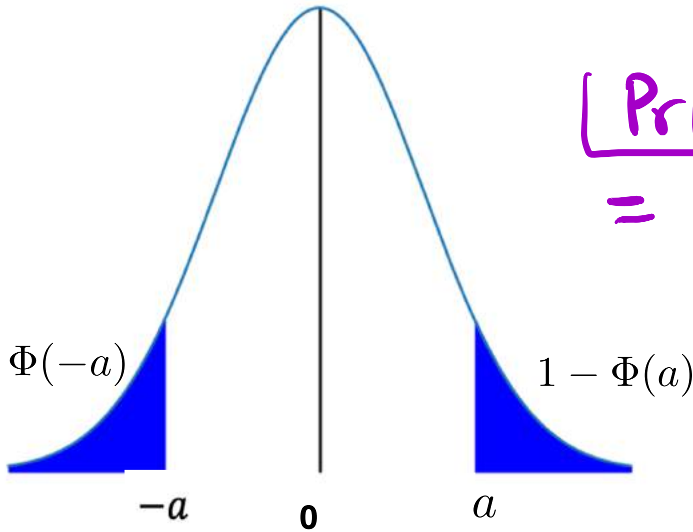


The Standard Normal CDF

$\Phi(a) = \Pr(Z \le a)$, $Z \sim \mathcal{N}(0,1)$

If $Z \sim \mathcal{N}(0,1)$, we denote the CDF $\Phi(a) = F_Z(a) = P(Z \le a)$, since it's so commonly used. There is no closed-form formula, so this CDF is stored in a Φ table.

$$\Phi(-a) = 1 - \Phi(a)$$



$\Pr(Z > a) = 1 - \Phi(a)$

$= \Pr(Z < -a)$

$\Phi(-a)$

The Standard Normal CDF

$$P(Z \le 1.09) = \Phi(1.09) \approx 0.8621$$

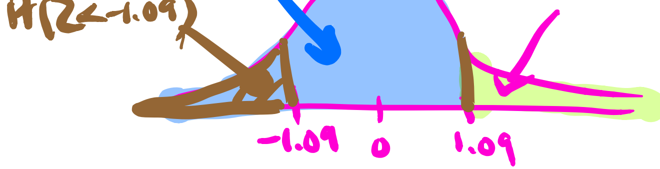
N(0,1)

Φ Table: $\mathbb{P}(Z \le z)$ when $Z \sim \mathcal{N}(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

Pr(Z ≤ 1.09)

Pr(Z > 1.09)



← pdf for N(0,1)

$1-0.8621 = 0.1379$

The Standard Normal CDF

$$P(Z \le 1.09) = \Phi(1.09) \approx \underline{0.8621}$$

$= 1-\Pr(Z\le 1.09)$

	Pr(Z > 1.09)	Pr(Z < −1.09)
a)	0.8621	0.1379
b)	0.1379	0.8621
c)	$\boxed{0.1379}$	$\boxed{0.1379}$
d)	I don't know	

Φ Table: $\mathbb{P}(Z \le z)$ when $Z \sim \mathcal{N}(0,1)$

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2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

What about non-Standard Normals?

$X \sim \mathcal{N}(\mu, \sigma^2)$,

$$\boxed{W = X - \mu}$$

$$E(W) = E(X) - E(\mu)$$
$$= \mu - \mu = 0$$

	$E(W)$	$Var(W)$
a)	0	σ^2
b)	μ	σ^2
c)	0	1
d)	I don't know	

$$\boxed{Y = \frac{W}{\sigma}}$$

$a = \frac{1}{\sigma}$

$$Var(Y) = \frac{Var(W)}{\sigma^2}$$
$$= 1$$

$Var(aX) = a^2 \, Var(X)$

	$E(Y)$	$Var(Y)$
a)	$\frac{\mu}{\sigma}$	σ
b)	0	1
c)	0	σ^2
d)	I don't know	

X with mean μ, var σ^2

$\boxed{\frac{X-\mu}{\sigma}}$ has mean 0 var 1

We can Standardize any RV

Let X be **ANY** random variable (discrete or continuous) with $E[X] = \mu$ and $Var(X) = \sigma^2$, and $a, b \in \mathbb{R}$. Then,

$$E[aX + b] = aE[X] + b = a\mu + b$$

$$Var(aX + b) = a^2 Var(X) = a^2\sigma^2$$

In particular, we call $\frac{X-\mu}{\sigma}$ a standardized version of X, as it measures how many standard deviations above the mean a point is.

$$E\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma}(E[X] - \mu) = 0$$

$$Var\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2} Var(X - \mu) = \frac{1}{\sigma^2}\sigma^2 = 1$$

$a = \frac{1}{\sigma}$

$b = -\frac{\mu}{\sigma}$

true for any r.v.

NORMALS STAY NORMAL! (UNDER SCALE + SHIFT)

$X \sim N(\mu, \sigma^2)$

$\boxed{aX+b}$ still normal.

$$\boxed{aX+b \sim N(a\mu+b,\ a^2\sigma^2)}$$

$aX+b$	mean	variance
a)	$a\mu$	$a\sigma^2$
b)	$a\mu$	$a^2\sigma^2$
c)	$a\mu+b$	$a^2\sigma^2+b^2$
d)	$a\mu+b$	$a^2\sigma^2$

$X \sim Geo(p)$

$\boxed{5X}$ 5, 10, 15, 20, ...

Closure of the Normal (Under Scale+Shift)

Let X be **ANY** random variable (discrete or continuous) with $E[X] = \mu$ and $Var(X) = \sigma^2$, and $a, b \in \mathbb{R}$. Recall,

$$E[aX + b] = aE[X] + b = a\mu + b$$

$$Var(aX + b) = a^2 Var(X) = a^2\sigma^2$$

But if $X \sim \mathcal{N}(\mu, \sigma^2)$ (a Normal rv), then

$$aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$

In particular,

$$\frac{X - \mu}{\sigma} \sim \mathcal{N}(0,1)$$

Note the "special" thing here is that the transformed RV remains a Normal rv - the mean and variance are no surprise.

$\frac{X-\mu}{\sigma} \sim N(0,1)$

$$\boxed{X \sim N(3, 16)}$$

$$\Downarrow$$

$$\sigma = 4$$

X is normal with mean 3 and variance 16.

What is

- Pr (2 < X < 5)

$$= Pr\left(\frac{2-3}{4} < \boxed{\frac{X-3}{4}} < \frac{5-3}{4}\right)$$

$$= Pr\left(-\frac{1}{4} < \underline{Z} < \frac{1}{2}\right)$$

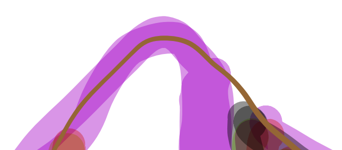
$$= \Phi\left(\tfrac{1}{2}\right) - \boxed{\Phi\left(-\tfrac{1}{4}\right)}$$

$$= \Phi\left(\tfrac{1}{2}\right) - \boxed{\left[1 - \Phi\left(\tfrac{1}{4}\right)\right]}$$

$$= 0.69146 - 1 + 0.58871$$

- Pr (X > 0)

- Pr (|X−3| > 6)



$=$

$= 1 - \Pr(Z \le \frac{1}{4})$

X is normal with mean 3 and variance 16

What is

- Pr (2 < X < 5)

- Pr (X > 0)

$= \Pr\left(\frac{X-3}{4} > \frac{0-3}{4}\right)$

$= \Pr(Z > -\frac{3}{4}) = \Pr(Z < \frac{3}{4}) = \Phi(\frac{3}{4})$

$= 0.77337$

- Pr (|X-3| > 6)

$= \Pr(X > 9) + \Pr(X < -3)$

$= \Pr\left(\frac{X-3}{4} > \frac{9-3}{4}\right) + \Pr\left(\frac{X-3}{4} < \frac{-3-3}{4}\right)$

$= \Pr(Z > 1.5) + \Pr(Z < -1.5)$

$= 2\Pr(Z > 1.5) = 2(1 - \Pr(Z \le 1.5)) = 2(1 - \Phi(1.5))$

$= 2(1 - 0.93319)$

Φ Table: $\mathbb{P}(Z \le z)$ when $Z \sim \mathcal{N}(0,1)$

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2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

FROM $N(\mu, \sigma^2)$ TO STANDARD NORMAL

For a $X \sim \mathcal{N}(\mu, \sigma^2)$, we have

$$F_X(y) = P(X \leq y) = P\left(\frac{X-\mu}{\sigma} \leq \frac{y-\mu}{\sigma}\right) = P\left(Z \leq \frac{y-\mu}{\sigma}\right) = \Phi\left(\frac{y-\mu}{\sigma}\right)$$

$$P(a \leq X \leq b) = F_X(b) - F_X(a) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

Summary: The Normal/Gaussian RV

Normal (Gaussian, "bell curve") Distribution: $X \sim \mathcal{N}(\mu, \sigma^2)$ if and only if X has the following pdf:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$E[X] = \mu \qquad\qquad Var(X) = \sigma^2$$

The "standard normal" random variable is typically denoted Z and has mean 0 and variance 1. By the closure property of normals, if $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$. The CDF has no closed form, but we denote the CDF of the standard normal by $\Phi(a) = F_Z(a) = P(Z \le a)$. Note that by symmetry of the density about 0, $\Phi(-a) = 1 - \Phi(a)$.

Closure of the Normal (Under Addition)

$X \sim N(\mu, \sigma^2)$

$aX + b$ also normal.

$X \sim N(\mu_x, \sigma_x^2)$ $\quad$ $Y \sim N(\mu_y, \sigma_y^2)$

$$\boxed{aX + bY} \sim N(a\mu_x + b\mu_y, a^2\sigma_x^2 + b^2\sigma_y^2)$$

X & Y independent.

Closure of the normal (under addition)

Let X, Y be **ANY independent** random variables (discrete or continuous) with $E[X] = \mu_X$, $E[Y] = \mu_Y$, $Var(X) = \sigma_X^2$, $Var(Y) = \sigma_Y^2$, and $a, b, c \in \mathbb{R}$. Recall,

$$E[aX + bY + c] = aE[X] + bE[Y] + c = a\mu_X + b\mu_Y + c$$

$$Var(aX + bY + c) = a^2 Var(X) + b^2 Var(Y) = a^2\sigma_X^2 + b^2\sigma_Y^2$$

But if $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ (both independent Normal rvs), then

$$aX + bY + c \sim \mathcal{N}(a\mu_X + b\mu_Y + c, a^2\sigma_X^2 + b^2\sigma_Y^2)$$

Note the "special" thing here is that the sum remains a Normal rv – the mean and variance are no surprise.

5.7 The Central Limit Theorem

The Sample Mean

Sample Mean: Let $X_1, X_2, \ldots X_n$ be a sequence of iid random variables with mean μ and variance σ^2. The sample mean is

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$Var(aX) = a^2 Var(X)$

$$E[\bar{X}_n] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n} n\mu = \mu$$

$$Var(\bar{X}_n) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \overset{\text{indep.}}{=} \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$$

iid $\equiv$ independent, identically distributed

The Central Limit Theorem

arbitrary (discrete or cont)

Consider i.i.d. (independent, identically distributed) random vars $X_1, X_2, X_3, \ldots X_n$

Where X_i has $\mu = E[X_i]$ and $\sigma^2 = Var[X_i]$

Consider random variables

$$X_1 + X_2 + \ldots + X_n$$

and

$$\boxed{\frac{1}{n}\sum_{i=1}^{n} X_i}$$

mean	variance
$n\mu$	$n\sigma^2$
μ	$\frac{\sigma^2}{n}$

The Central Limit Theorem

X_i pdf $\boxed{f}$ pmf p_X

arbitrary distn

Consider i.i.d. (independent, identically distributed) random vars
$X_1, X_2, X_3, \ldots$
Where X_i has $\mu = E[X_i]$ and $\sigma^2 = Var[X_i]$

finite mean variance

As $n \to \infty$,

$$\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \to N(0,1)$$

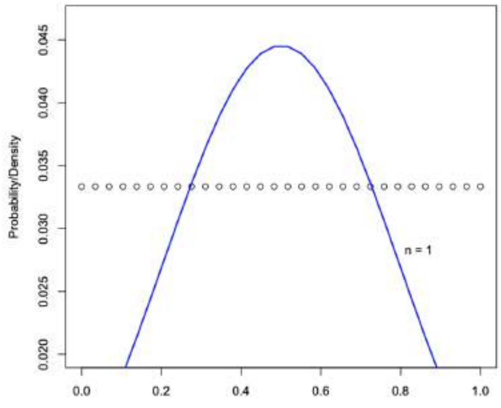
standardized version.

Restated: As $n \to \infty$,

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to N\left(\mu, \frac{\sigma^2}{n}\right)$$

Sample mean version.

CLT (Pictures)



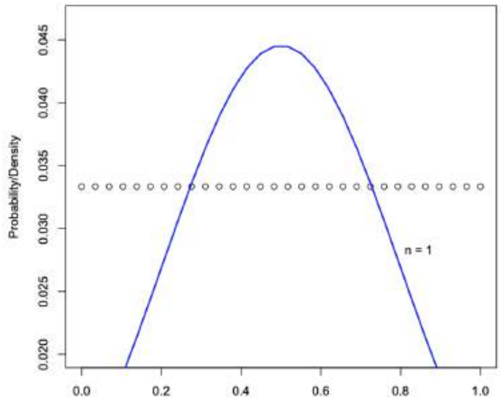
$X_1 \sim \text{Unif}(\mu, \sigma^2$

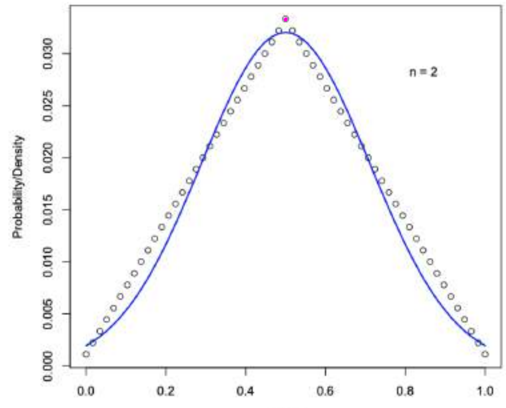
$Z(\mu, \sigma^2)$

CLT (Pictures)

$X_1 + X_2$

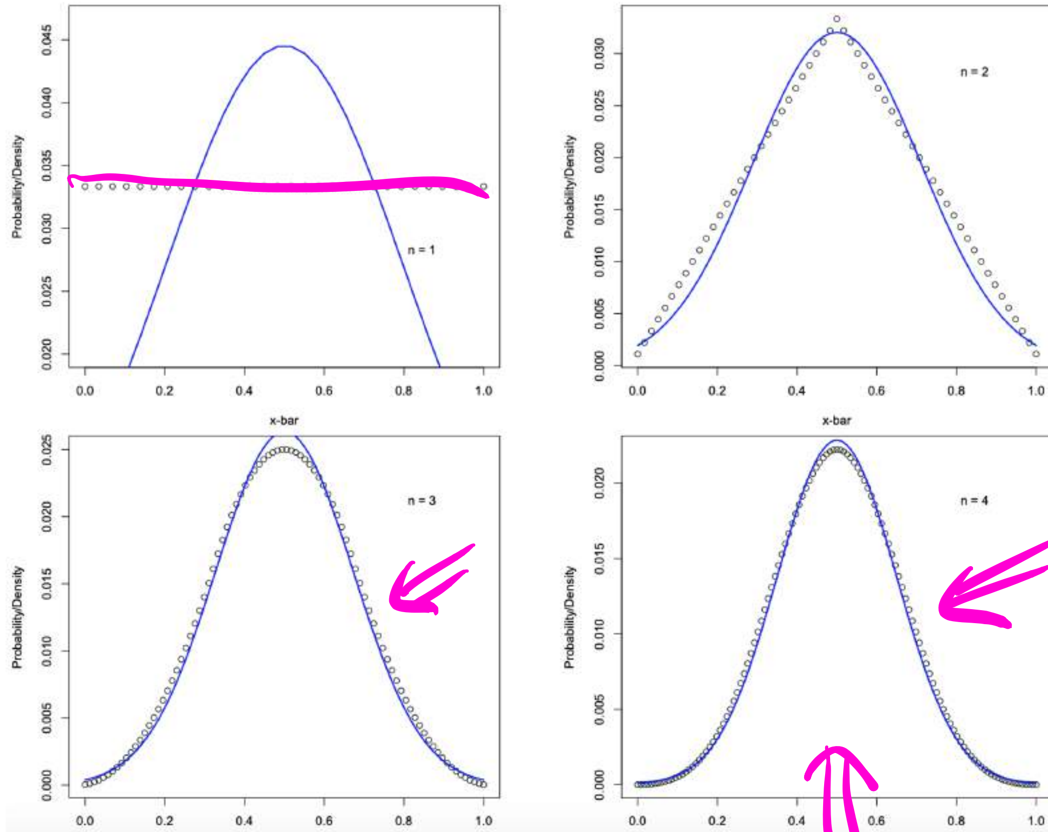
b/c uniform





$X_1 + X_2 + X_3 + X_4$

CLT (Pictures)

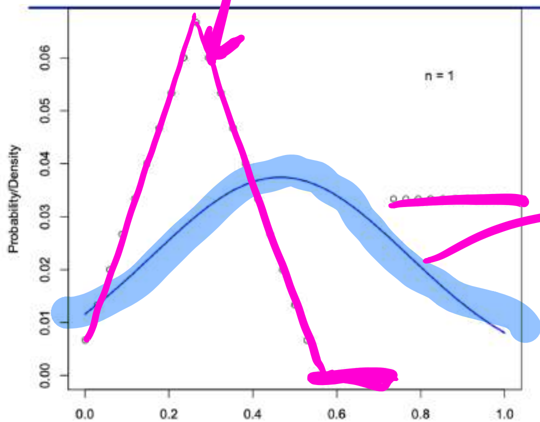


From: https://courses.cs.washington.edu/courses/cse312/17wi/slides/10limits.pdf

$$\frac{X_1 + X_2 + X_3 + X_4}{4}$$

CLT (Pictures)

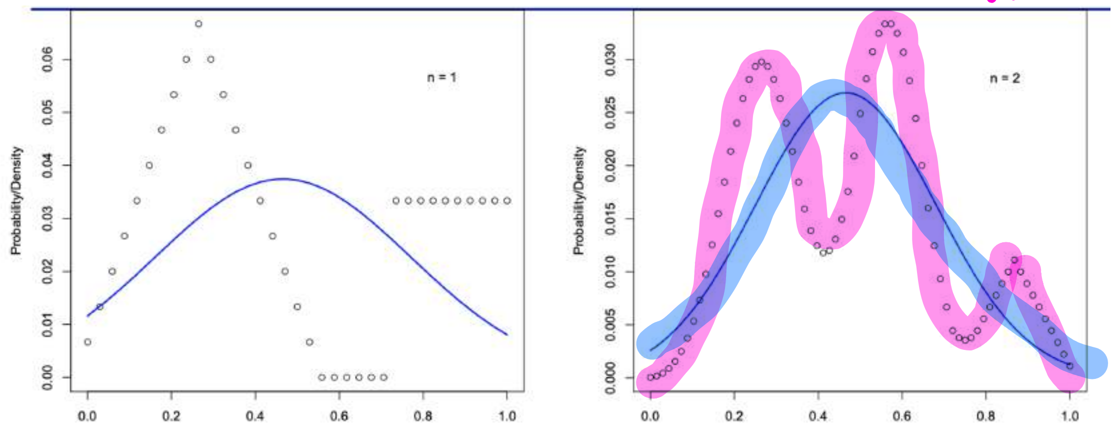
X crazy dist μ, σ^2



$N(\mu, \sigma^2)$

CLT (Pictures)

$\frac{X_1+X_2}{2}$ iid from wg



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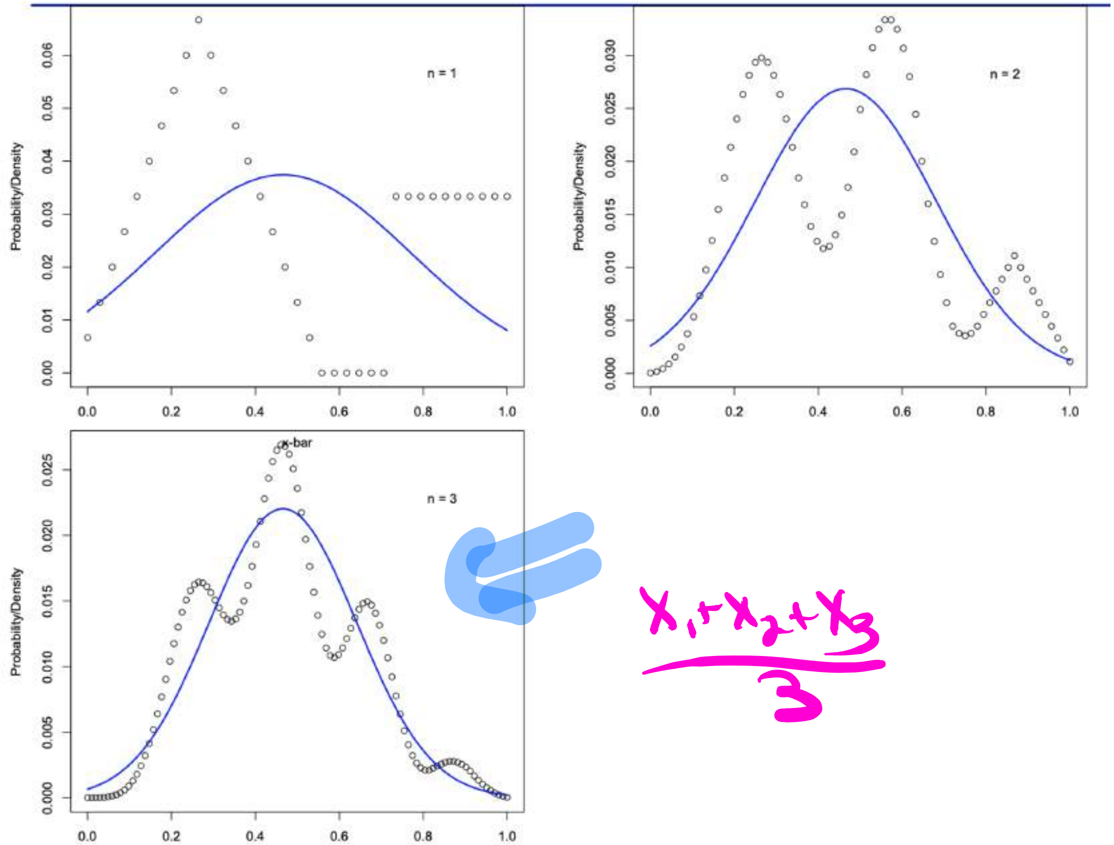
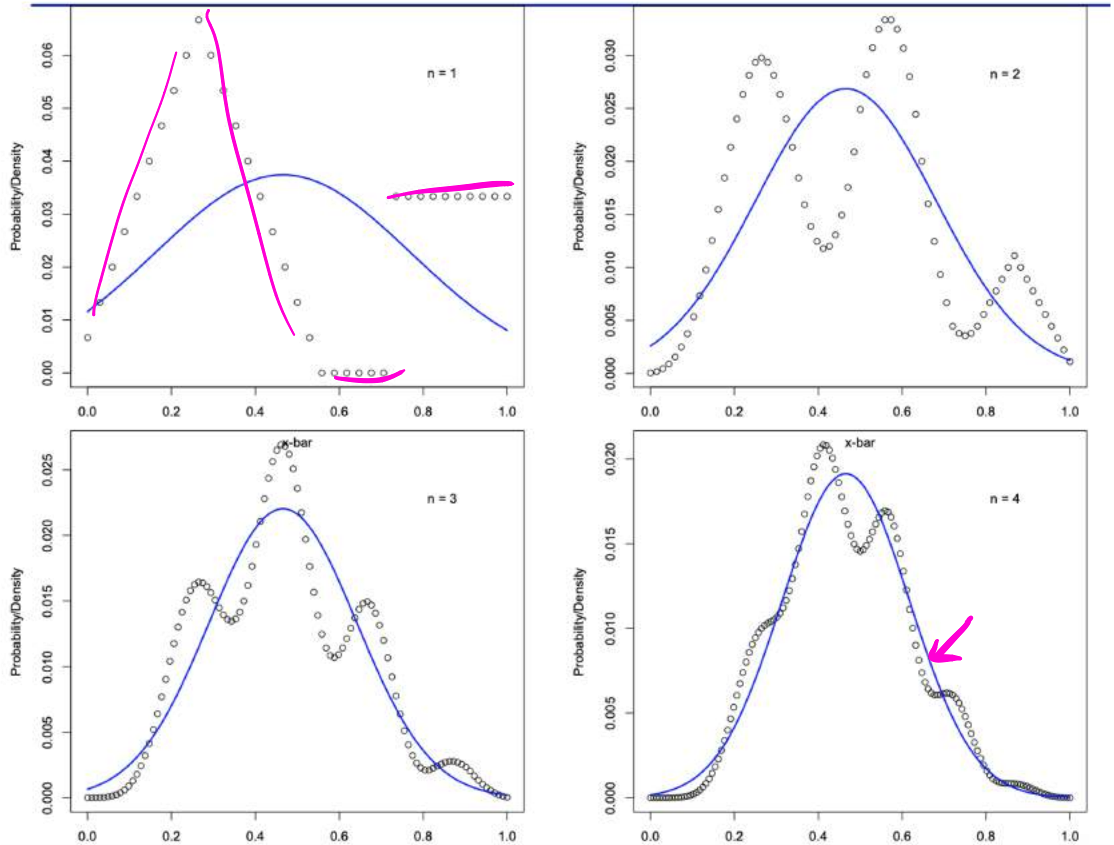


$$\frac{X_1+X_2+X_3}{3}$$

From: https://courses.cs.washington.edu/courses/cse312/17wi/slides/10limits.pdf

CLT (Pictures)



From: https://courses.cs.washington.edu/courses/cse312/17wi/slides/10limits.pdf

$E(X_i) = \mu$

$Var(X_i) = \sigma^2$

$\frac{X_1 + X_2 + X_3 + X_4}{4}$

$\mu, \sigma^2 \in N$

CLT IN THE REAL WORLD

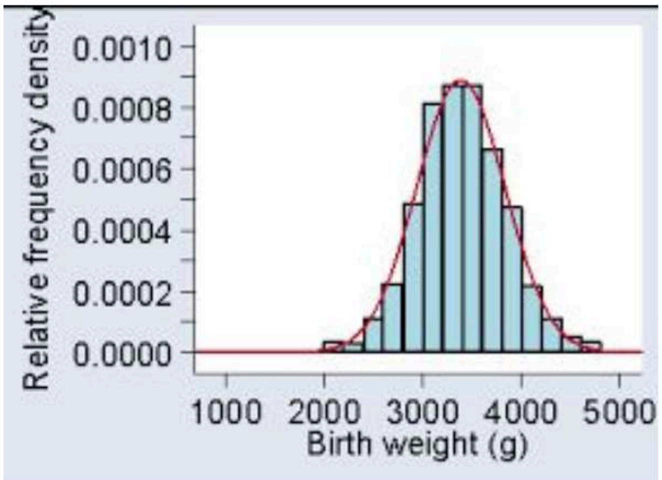
CLT is the reason many things appear normally distributed
Many quantities = sums of (roughly) independent random vars

Exam scores: sums of individual problems

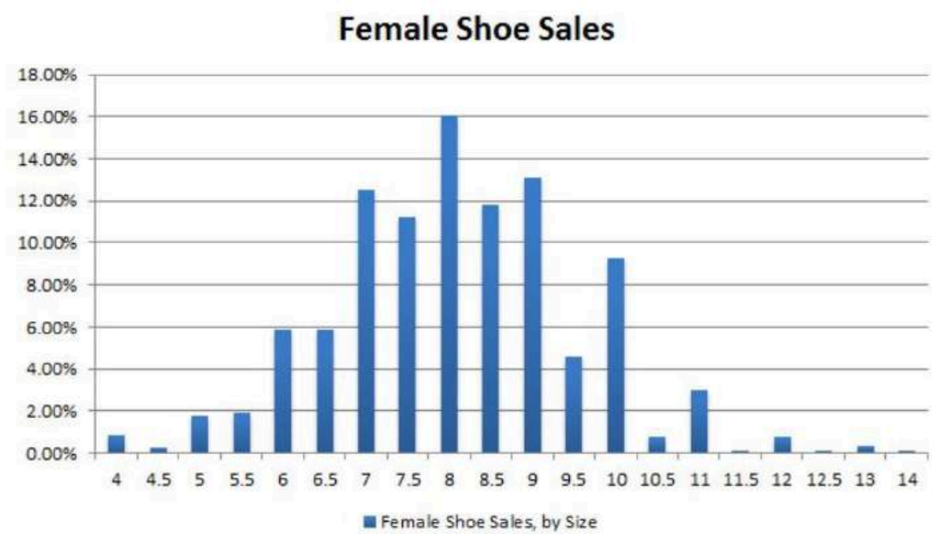
People's heights: sum of many genetic & environmental factors

Measurements: sums of various small instrument errors

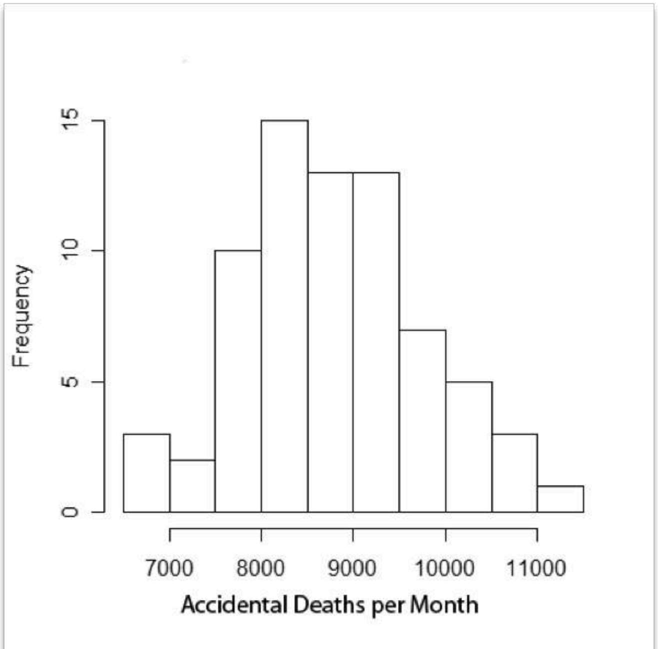
CLT in the real world



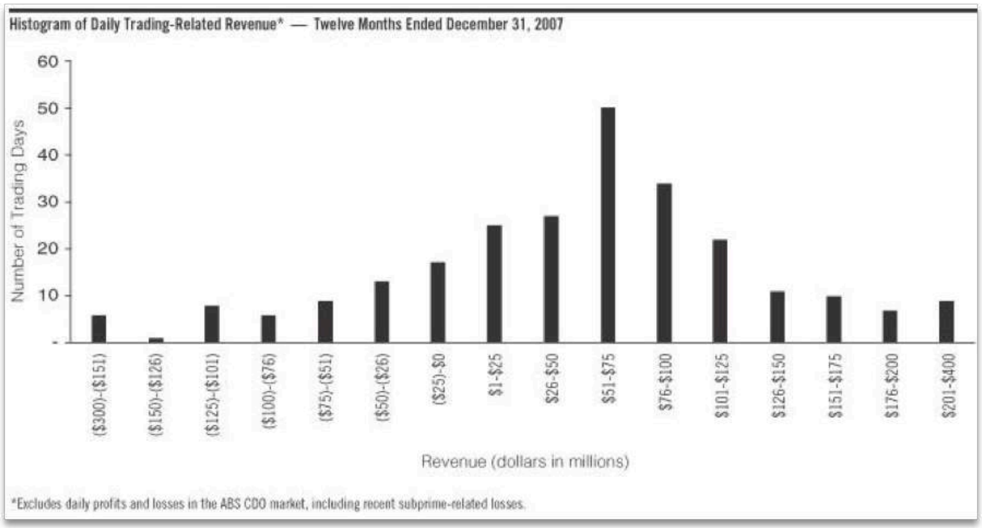
CLT in the real world



CLT in the real world



CLT in the real world



CLT in the real world