5.2 Joint Continuous Distributions

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Most slides by Alex Tsun

RECAP

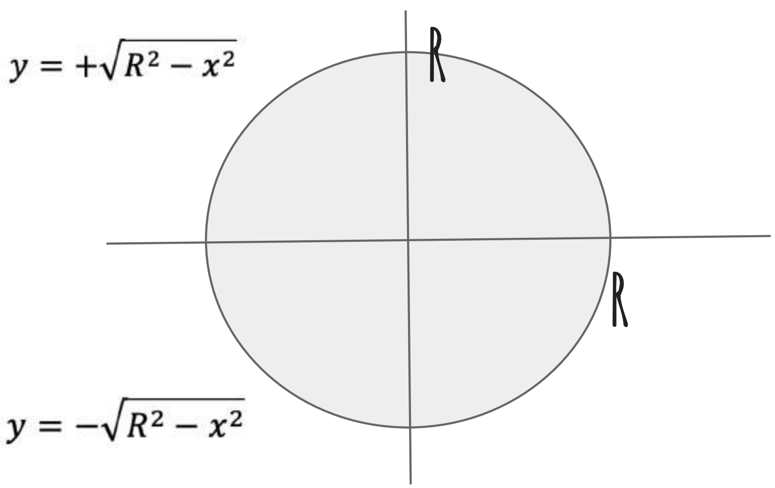
	Discrete	Continuous
Joint PMF/PDF	$p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$	$f_{X,Y}(x,y) \neq \mathbb{P}(X = x, Y = y)$
Joint range/support $\Omega_{X,Y}$	$\{(x,y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x,y) > 0\}$	$\{(x,y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x,y) > 0\}$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t \le x, s \le y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\, dsdt$
Normalization	$\sum_{x,y} p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dxdy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$
Expectation	$\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y) p_{X,Y}(x,y)$	$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dxdy$

Joint PDFs (Example 1)

Suppose (X, Y) are jointly and uniformly distributed on the circle of radius R centered at the origin (e.g., a dart throw). Set up but **DO NOT EVALUATE** any of your answers. Take care in setting up the limits of integration.

a. Find and sketch the joint range $\Omega_{X,Y}$.

$$\Omega_{X,Y} = \{(x, y) \in \mathbb{R}^2 : \ x^2 + y^2 \leq R^2\}$$



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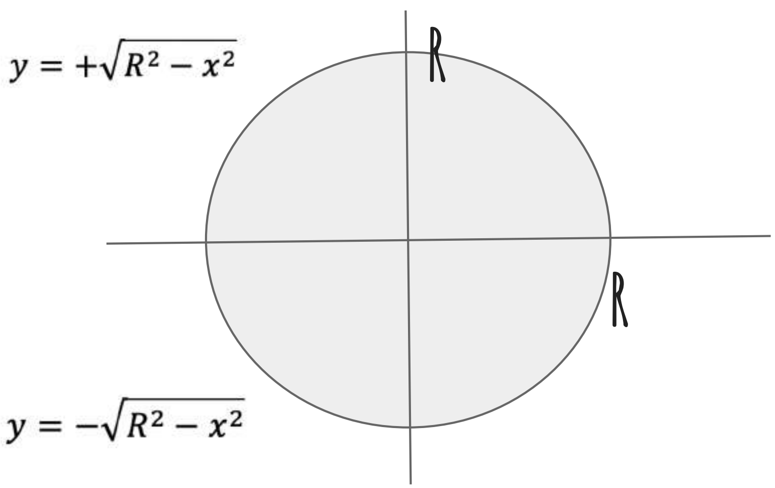

b. Write an expression for the joint PDF $f_{X,Y}(x, y)$ and carefully define it for all $x, y \in \mathbb{R}$.

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\pi R^2}, & x, y \in \Omega_{X,Y} \\ 0, & \text{otherwise} \end{cases}$$

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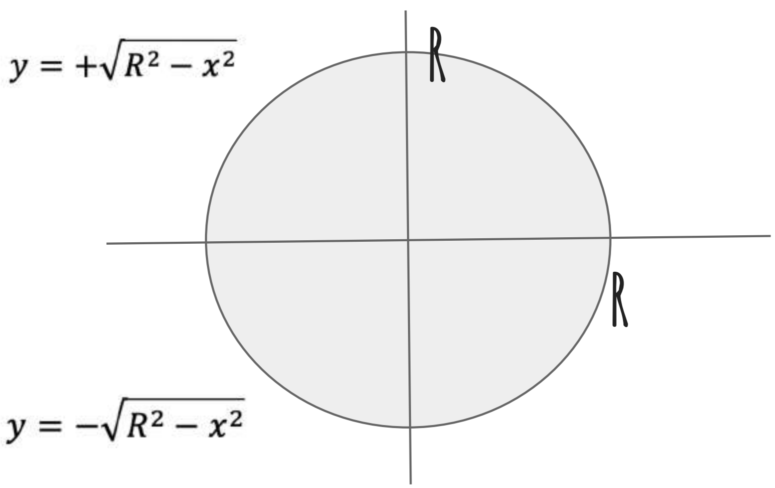

c. Find Ω_X and write an expression that we could evaluate to find $f_X(x)$.

$$\Omega_X = [-R, R]$$

$$f_X(x) = \int_{-\sqrt{R^2-x^2}}^{+\sqrt{R^2-x^2}} f_{X,Y}(x, y)\,dy$$

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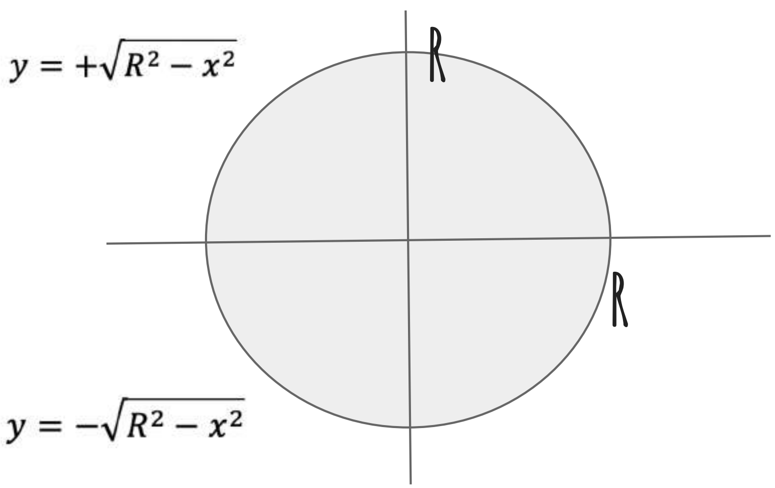

d. Let Z the distance from the center that the dart falls. Find Ω_Z and write an expression for $E[Z]$.

$$Z = \sqrt{X^2 + Y^2}$$
$$\Omega_Z = [0, R]$$

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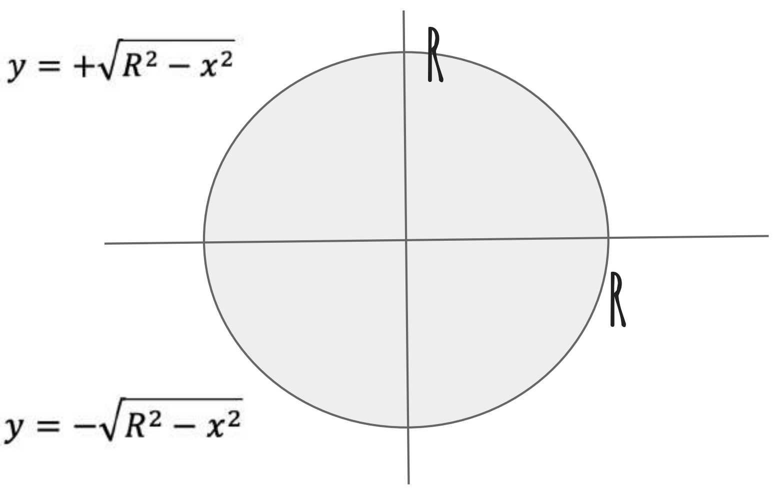

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$$Z = \sqrt{X^2 + Y^2}$$

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$$E[Z] = \int_{-R}^{R} \int_{-\sqrt{R^2 - x^2}}^{+\sqrt{R^2 - x^2}} \sqrt{x^2 + y^2} f_{X,Y}(x, y) dy dx$$

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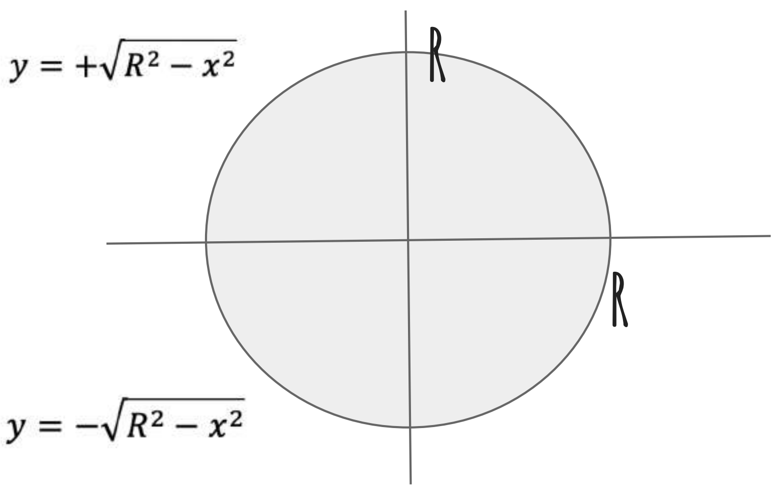

e. Are X and Y independent?

No, $\Omega_{X,Y} \neq \Omega_X \times \Omega_Y$ (not a rectangle).

$$0 = f_{X,Y}(0.99R, 0.99R) \neq f_X(0.99R) f_Y(0.99R) > 0$$

Random Picture

Joint PDFs (Example 2)

Consider a continuous joint distribution, (X, Y), where $X \in [0,1]$ is the proportion of the time until the midterm that you actually study for it, and $Y \in [0,1]$ is your percentage score on the exam. Set up but **DO NOT EVALUATE** any of your answers. Take care in setting up the limits of integration. The joint PDF is:

$$f_{X,Y}(x,y) = \begin{cases} ce^{-(y-x)}, & x,y \in [0,1] \text{ and } y \geq x \\ 0, & y < x \end{cases}$$

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a. Sketch the joint range $\Omega_{X,Y}$, and interpret it in English.

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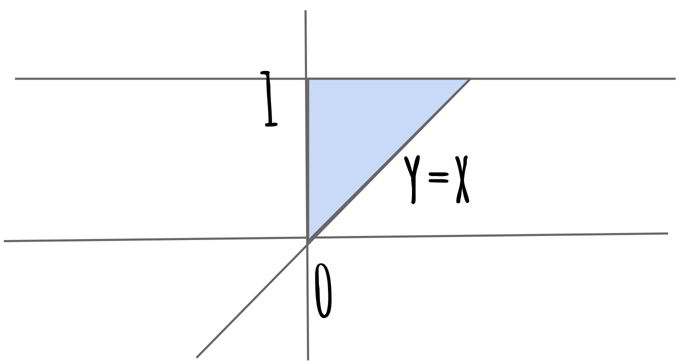
a. Sketch the joint range $\Omega_{X,Y}$, and interpret it in English.

Your score is at least the proportion of the time you study.



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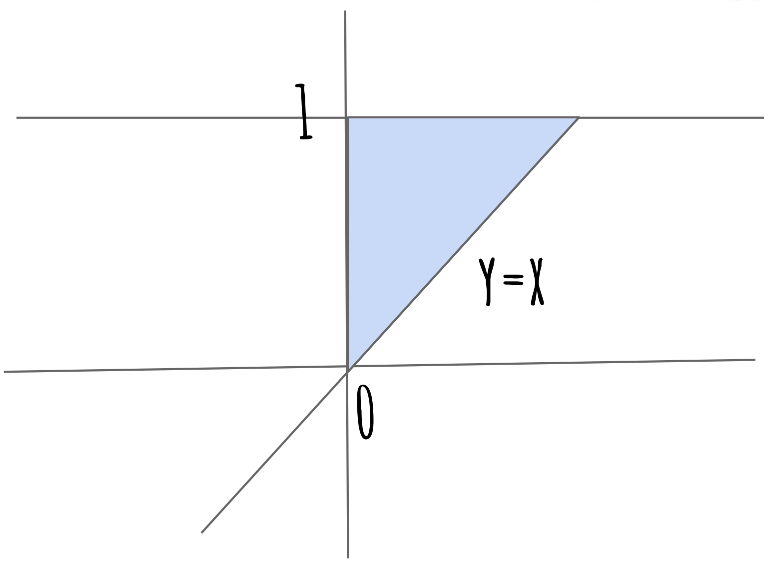

b. Write an expression that we could evaluate to find c.

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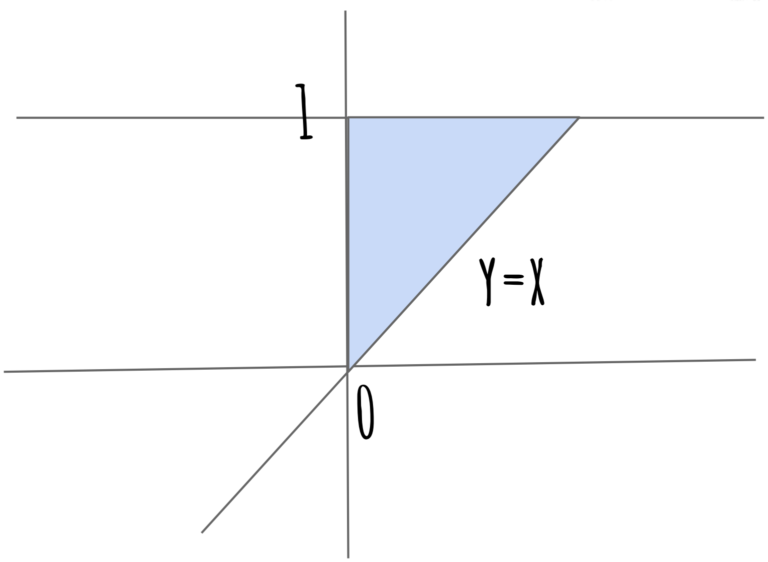

b. Write an expression that we could evaluate to find c.

$$c = \frac{1}{\int_0^1 \int_x^1 e^{-(y-x)} dy\, dx} = \frac{1}{\int_0^1 \int_0^y e^{-(y-x)} dx\, dy}$$

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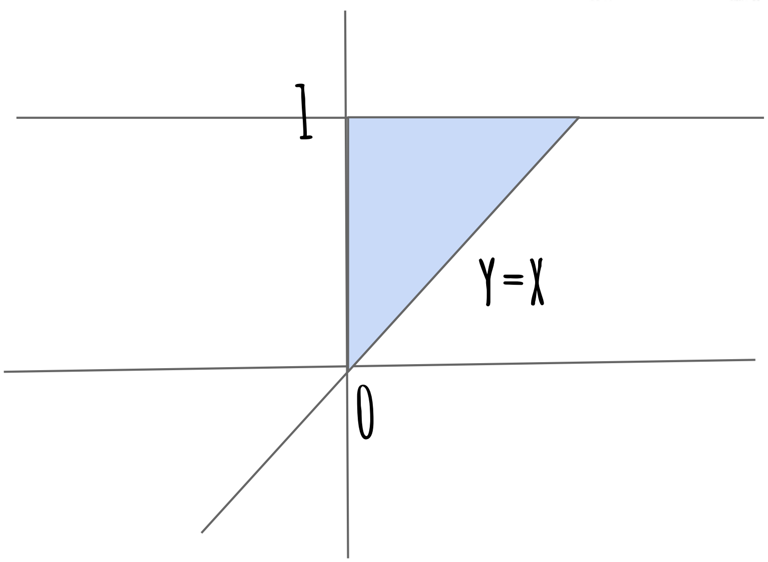

c. Find Ω_Y and write an expression that we could evaluate to find $f_Y(y)$.

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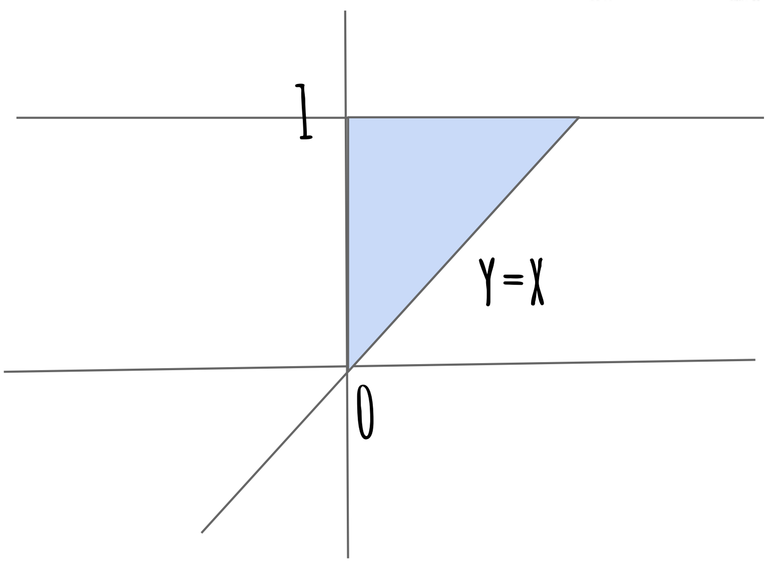

c. Find Ω_Y and write an expression that we could evaluate to find $f_Y(y)$.

$$\Omega_Y = [0,1]$$

$$f_Y(y) = \int_0^y ce^{-(y-x)}\,dx$$

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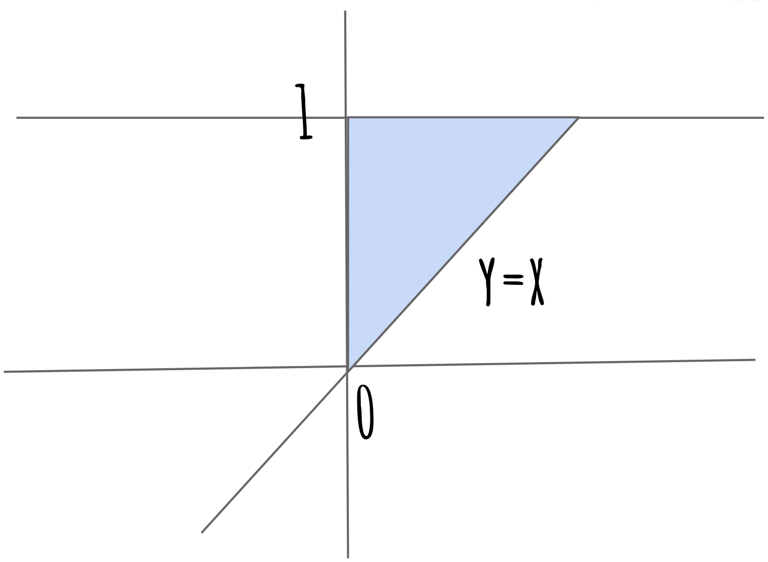

d. Write an expression that we could evaluate to find $P(Y \geq 0.9)$.

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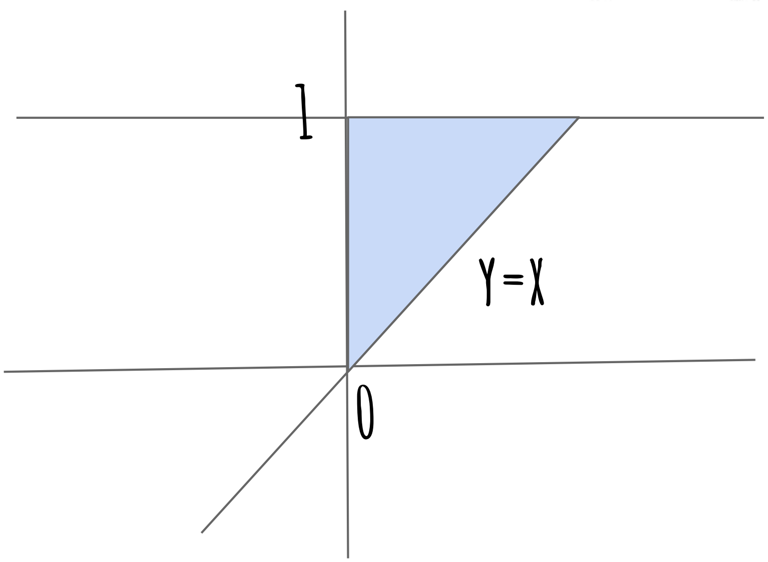

d. Write an expression that we could evaluate to find $P(Y \geq 0.9)$.

$$P(Y \geq 0.9) = \int_{0.9}^{1} f_Y(y)dy = \int_{0.9}^{1}\int_{0}^{y} ce^{-(y-x)}dx\,dy$$

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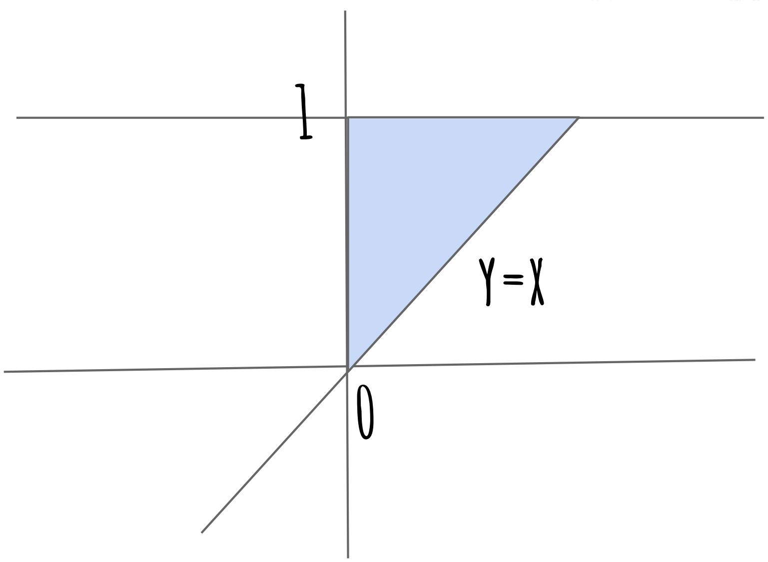

e. Write an expression that we could evaluate to find $E[Y]$.

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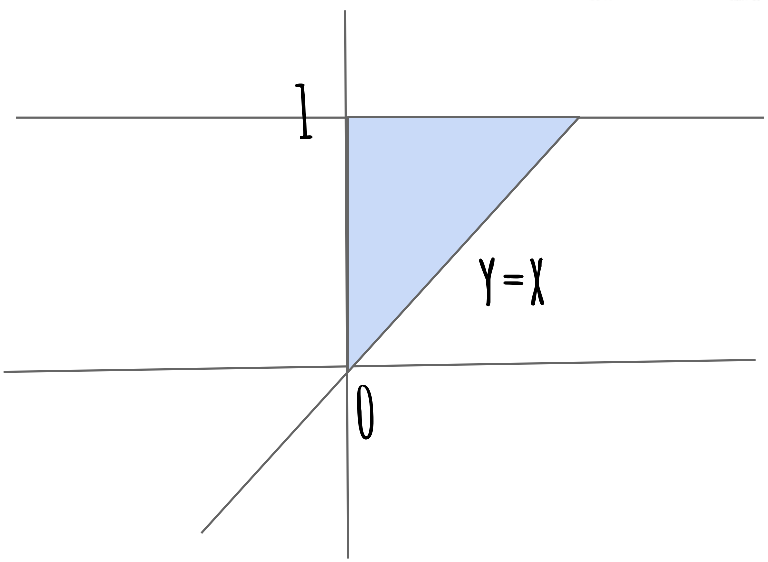

e. Write an expression that we could evaluate to find $E[Y]$.

$$E[Y] = \int_0^1 y f_Y(y) dy = \int_0^1 \int_0^y cye^{-(y-x)} dx\, dy$$

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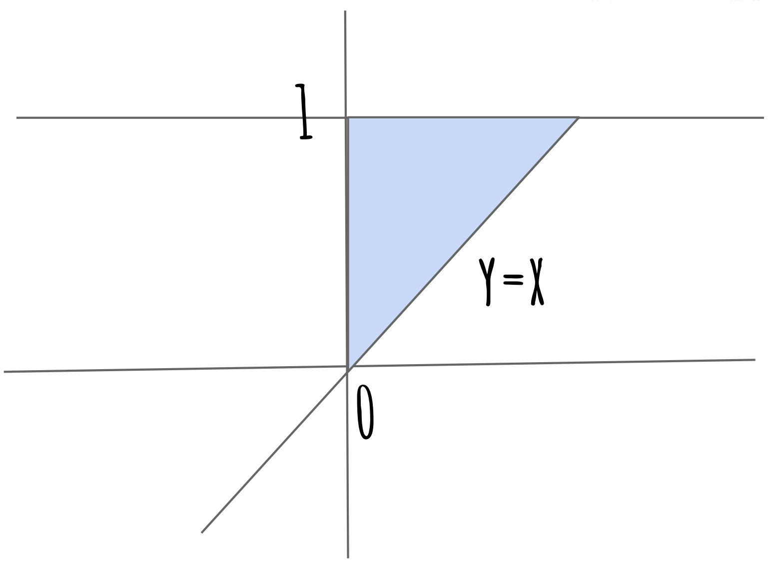

f. Are X and Y independent?

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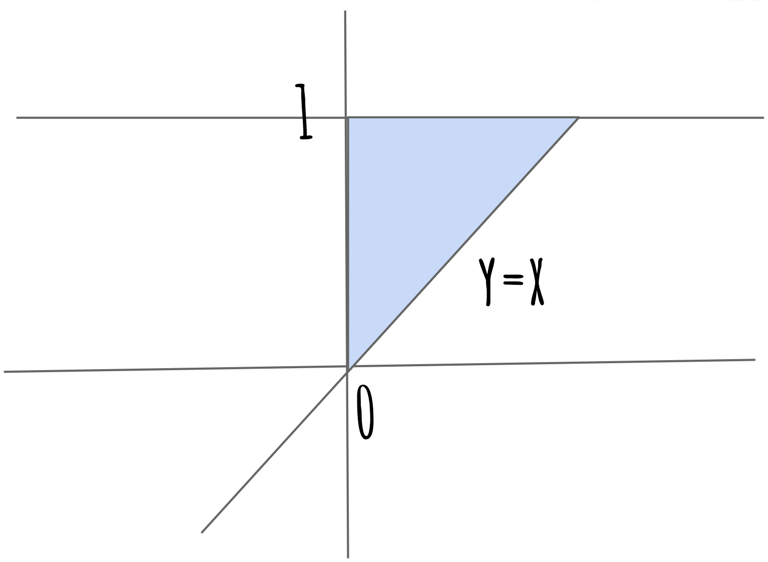

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No, $\Omega_{X,Y} \neq \Omega_X \times \Omega_Y$ (not a rectangle).

5.3 Law of Total Expectation

Agenda

- Conditional Expectation
- Law of Total Expectation (LTE)

Conditional Expectation

Conditional Expectation: Let X be a discrete random variable. Then, the conditional expectation of X given A is

$$\mathbb{E}[X \mid A] = \sum_{x \in \Omega_X} x\mathbb{P}(X = x|A)$$

Linearity of expectation still applies to conditional expectation: $\mathbb{E}[X + Y \mid A] = \mathbb{E}[X \mid A] + \mathbb{E}[Y \mid A]$

Law of Total Expectation

Law of Total Expectation (Event Version): Let X be a random variable, and let events $A_1, ..., A_n$ partition the sample space. Then,

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid A_i] \, \mathbb{P}(A_i)$$

Law of Total Expectation : Application

```
System that fails in step i independently
with probability p
X  # steps to fail
```

E(X) ?

Let A be the event that system fails in first step.

Law of Total Expectation : Application

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System that fails in step i independently
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E(X) ?

Let A be the event that system fails in first step.

$$E(X) = E(X|A)Pr(A) + E(X|\overline{A})Pr(\overline{A})$$

$$= p + (1 + E(X))(1 - p)$$

$$= 1 + (1 - p)E(X)$$

$$E(X) = \frac{1}{p}$$

Linearity of expectation applies

To conditional expectation too!!

$E(X+ Y \mid A) = E(X \mid A) + E(Y \mid A)$

$E(aX + b \mid A)= a\, E(X \mid A) + b$

Law of total Expectation (RV version)

Law of Total Expectation (RV Version): Suppose X and Y be discrete random variables. Then,

$$\mathbb{E}[X] = \sum_{y} \mathbb{E}[X \mid Y = y] \, p_Y(y)$$

Problem

The number of people who enter an elevator on the ground floor is a Poisson random variable with mean 10. If there are N floors above the ground floor, and if each person is equally likely to get off at any one of the N floors, independently of where the others get off, compute the expected number of stops that the elevator will make before discharging all the passengers.