5.2 Joint Continuous Distributions

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Most slides by Alex Tsun
## Recap

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<th>Discrete</th>
<th>Continuous</th>
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<tr>
<td><strong>Joint PMF/PDF</strong></td>
<td>$p_{X,Y}(x, y) = \mathbb{P}(X = x, Y = y)$</td>
<td>$f_{X,Y}(x, y) \neq \mathbb{P}(X = x, Y = y)$</td>
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<tr>
<td><strong>Joint range/support</strong></td>
<td>${(x, y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x, y) &gt; 0}$</td>
<td>${(x, y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x, y) &gt; 0}$</td>
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<tr>
<td><strong>Joint CDF</strong></td>
<td>$F_{X,Y}(x, y) = \sum_{t \leq x, s \leq y} p_{X,Y}(t, s)$</td>
<td>$F_{X,Y}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t, s) , ds , dt$</td>
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<tr>
<td><strong>Normalization</strong></td>
<td>$\sum_{x, y} p_{X,Y}(x, y) = 1$</td>
<td>$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) , dx , dy = 1$</td>
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<tr>
<td><strong>Marginal PMF/PDF</strong></td>
<td>$p_X(x) = \sum_{y} p_{X,Y}(x, y)$</td>
<td>$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) , dy$</td>
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<tr>
<td><strong>Expectation</strong></td>
<td>$\mathbb{E}[g(X, Y)] = \sum_{x, y} g(x, y) p_{X,Y}(x, y)$</td>
<td>$\mathbb{E}[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) , dx , dy$</td>
</tr>
</tbody>
</table>
JOINT PDFS (EXAMPLE 1)

Suppose $(X, Y)$ are jointly and uniformly distributed on the circle of radius $R$ centered at the origin (e.g., a dart throw). Set up but **DO NOT EVALUATE** any of your answers. Take care in setting up the limits of integration.

a. Find and sketch the joint range $\Omega_{X,Y}$.

$$\Omega_{X,Y} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq R^2\}$$

\[
y = +\sqrt{R^2 - x^2}
\]

\[
y = -\sqrt{R^2 - x^2}
\]
**Joint PDFs (Example 1)**

Suppose \((X, Y)\) are jointly and uniformly distributed on the circle of radius \(R\) centered at the origin (e.g., a dart throw). Set up but **DO NOT EVALUATE** any of your answers. Take care in setting up the limits of integration.

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\[\Omega_{X,Y} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq R^2\}\]

\[
y = +\sqrt{R^2 - x^2}
\]

b. Write an expression for the joint PDF \(f_{X,Y}(x, y)\) and carefully define it for all \(x, y \in \mathbb{R}\).

\[f_{X,Y}(x, y) = \begin{cases} 
\frac{1}{\pi R^2}, & x, y \in \Omega_{X,Y} \\
0, & \text{otherwise}
\end{cases}\]
**Joint PDFs (Example 1)**

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\]

\[
y = +\sqrt{R^2 - x^2}
\]

\[
y = -\sqrt{R^2 - x^2}
\]

c. Find \(\Omega_X\) and write an expression that we could evaluate to find \(f_X(x)\).

\[
\Omega_X = [-R, R]
\]

\[
f_X(x) = \int_{-\sqrt{R^2 - x^2}}^{+\sqrt{R^2 - x^2}} f_{X,Y}(x, y) \, dy
\]
**Joint PDFs (Example 1)**

Suppose $(X, Y)$ are jointly and uniformly distributed on the circle of radius $R$ centered at the origin (e.g., a dart throw). Set up but **DO NOT EVALUATE** any of your answers. Take care in setting up the limits of integration.

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\[ y = +\sqrt{R^2 - x^2} \]

\[ y = -\sqrt{R^2 - x^2} \]

\[ R \]

\[ R \]

d. Let $Z$ the distance from the center that the dart falls. Find $\Omega_Z$ and write an expression for $E[Z]$.

$$Z = \sqrt{X^2 + Y^2}$$

$$\Omega_Z = [0, R]$$
**Joint PDFs (Example 1)**

Suppose \((X, Y)\) are jointly and uniformly distributed on the circle of radius \(R\) centered at the origin (e.g., a dart throw). Set up but **DO NOT EVALUATE** any of your answers. Take care in setting up the limits of integration.

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b. Let \(Z\) the distance from the center that the dart falls. Find \(\Omega_Z\) and write an expression for \(E[Z]\).

\[
Z = \sqrt{X^2 + Y^2}
\]

\[
\Omega_Z = [0, R]
\]

e. \[E[Z] = \int_{-R}^{R} \int_{-\sqrt{R^2 - x^2}}^{\sqrt{R^2 - x^2}} \sqrt{x^2 + y^2} f_{X,Y}(x, y) dy dx\]
**JOINT PDFs (Example 1)**

Suppose \((X, Y)\) are jointly and uniformly distributed on the circle of radius \(R\) centered at the origin (e.g., a dart throw). Set up but **DO NOT EVALUATE** any of your answers. Take care in setting up the limits of integration.

a. Find and sketch the joint range \(\Omega_{X,Y}\).

\[ \Omega_{X,Y} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq R^2\} \]

\[
y = +\sqrt{R^2 - x^2} \quad R
\]

\[
y = -\sqrt{R^2 - x^2} \quad R
\]

e. Are \(X\) and \(Y\) independent?

**No, \(\Omega_{X,Y} \neq \Omega_X \times \Omega_Y\) (not a rectangle).**

\[ 0 = f_{X,Y}(0.99R, 0.99R) \neq f_X(0.99R)f_Y(0.99R) > 0 \]
Random Picture
Joint PDFs (Example 2)

Consider a continuous joint distribution, \((X, Y)\), where \(X \in [0,1]\) is the proportion of the time until the midterm that you actually study for it, and \(Y \in [0,1]\) is your percentage score on the exam. Set up but **DO NOT EVALUATE** any of your answers. Take care in setting up the limits of integration. The joint PDF is:

\[
f_{X,Y}(x, y) = \begin{cases} 
  ce^{-(y-x)}, & x, y \in [0,1] \text{ and } y \geq x \\
  0, & y < x
\end{cases}
\]
JOINT PDFS (EXAMPLE 2)

Consider a continuous joint distribution, $(X, Y)$, where $X \in [0,1]$ is the proportion of the time until the midterm that you actually study for it, and $Y \in [0,1]$ is your percentage score on the exam. Set up but **DO NOT EVALUATE** any of your answers. Take care in setting up the limits of integration. The joint PDF is:

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  ce^{-(y-x)}, & x, y \in [0,1] \text{ and } y \geq x \\
  0, & y < x
\end{cases}$$

a. Sketch the joint range $\Omega_{X,Y}$, and interpret it in English.
Joint PDFs (Example 2)

Consider a continuous joint distribution, \((X, Y)\), where \(X \in [0,1]\) is the proportion of the time until the midterm that you actually study for it, and \(Y \in [0,1]\) is your percentage score on the exam. Set up but DO NOT EVALUATE any of your answers. Take care in setting up the limits of integration. The joint PDF is:

\[
f_{X,Y}(x, y) = \begin{cases} \ce^{- (y-x)}, & x, y \in [0,1] \text{ and } y \geq x \\ 0, & y < x \end{cases}
\]

a. Sketch the joint range \(\Omega_{X,Y}\), and interpret it in English.

Your score is at least the proportion of the time you study.
**Joint PDFs (Example 2)**

Consider a continuous joint distribution, \((X, Y)\), where \(X \in [0,1]\) is the proportion of the time until the midterm that you actually study for it, and \(Y \in [0,1]\) is your percentage score on the exam. Set up but **DO NOT EVALUATE** any of your answers. Take care in setting up the limits of integration. The joint PDF is:

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  0, & y < x 
\end{cases}
\]

b. Write an expression that we could evaluate to find \(c\).
**Joint PDFs (Example 2)**

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  0, & y < x
\end{cases}
\]

b. Write an expression that we could evaluate to find \(c\).

\[
c = \frac{1}{\int_0^1 \int_x^1 e^{-(y-x)} \, dy \, dx} = \frac{1}{\int_0^1 \int_0^y e^{-(y-x)} \, dx \, dy}
\]
JOINT PDFs (Example 2)

Consider a continuous joint distribution, \((X, Y)\), where \(X \in [0,1]\) is the proportion of the time until the midterm that you actually study for it, and \(Y \in [0,1]\) is your percentage score on the exam. Set up but **DO NOT EVALUATE** any of your answers. Take care in setting up the limits of integration. The joint PDF is:

\[
 f_{X,Y}(x, y) = \begin{cases} 
  ce^{-(y-x)}, & x, y \in [0,1] \text{ and } y \geq x \\
  0, & y < x 
\end{cases}
\]

\[ \Omega_Y \]

\[ y = x \]

\[ 0 \]

\[ 1 \]

c. Find \(\Omega_Y\) and write an expression that we could evaluate to find \(f_Y(y)\).
Joint PDFs (Example 2)

Consider a continuous joint distribution, \((X, Y)\), where \(X \in [0,1]\) is the proportion of the time until the midterm that you actually study for it, and \(Y \in [0,1]\) is your percentage score on the exam. Set up but **DO NOT EVALUATE** any of your answers. Take care in setting up the limits of integration. The joint PDF is:

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f_{X,Y}(x, y) = \begin{cases} 
    ce^{-(y-x)}, & x, y \in [0,1] \text{ and } y \geq x \\
    0, & y < x 
\end{cases}
\]

\[
\Omega_Y = [0,1]
\]

\[
f_Y(y) = \int_0^y ce^{-(y-x)} \, dx
\]

c. Find \(\Omega_Y\) and write an expression that we could evaluate to find \(f_Y(y)\).
Consider a continuous joint distribution, $(X, Y)$, where $X \in [0,1]$ is the proportion of the time until the midterm that you actually study for it, and $Y \in [0,1]$ is your percentage score on the exam. Set up but **DO NOT EVALUATE** any of your answers. Take care in setting up the limits of integration. The joint PDF is:

$$f_{X,Y}(x, y) = \begin{cases} \frac{c e^{-(y-x)}}{x}, & x, y \in [0,1] \text{ and } y \geq x \\ 0, & y < x \end{cases}$$

\[d. \text{ Write an expression that we could evaluate to find } P(Y \geq 0.9).\]
**Joint PDFs (Example 2)**

Consider a continuous joint distribution, \((X, Y)\), where \(X \in [0,1]\) is the proportion of the time until the midterm that you actually study for it, and \(Y \in [0,1]\) is your percentage score on the exam. Set up but DO NOT EVALUATE any of your answers. Take care in setting up the limits of integration. The joint PDF is:

\[
f_{X,Y}(x, y) = \begin{cases} 
  ce^{-(y-x)}, & x, y \in [0,1] \text{ and } y \geq x \\
  0, & y < x 
\end{cases}
\]

![Graph of the joint PDF with the line \(y = x\) and shaded region.]

\[d.\] Write an expression that we could evaluate to find \(P(Y \geq 0.9)\).

\[
P(Y \geq 0.9) = \int_{0.9}^{1} f_Y(y) dy = \int_{0.9}^{1} \int_{0}^{y} ce^{-(y-x)} dx \, dy
\]
**Joint PDFs (Example 2)**

Consider a continuous joint distribution, \((X,Y)\), where \(X \in [0,1]\) is the proportion of the time until the midterm that you actually study for it, and \(Y \in [0,1]\) is your percentage score on the exam. Set up but **DO NOT EVALUATE** any of your answers. Take care in setting up the limits of integration. The joint PDF is:

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  ce^{-(y-x)}, & x, y \in [0,1] \text{ and } y \geq x \\
  0, & y < x 
\end{cases}
\]

**(e)** Write an expression that we could evaluate to find \(E[Y]\).
JOINT PDFs (Example 2)

Consider a continuous joint distribution, \((X, Y)\), where \(X \in [0,1]\) is the proportion of the time until the midterm that you actually study for it, and \(Y \in [0,1]\) is your percentage score on the exam. Set up but **DO NOT EVALUATE** any of your answers. Take care in setting up the limits of integration. The joint PDF is:

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  0, & y < x 
\end{cases}
\]

\[1\]
\[y=x\]
\[0\]

e. Write an expression that we could evaluate to find \(E[Y]\).

\[
E[Y] = \int_0^1 y f_Y(y) dy = \int_0^1 \int_0^y cy e^{-(y-x)} dx \, dy
\]
JOINT PDFs (Example 2)

Consider a continuous joint distribution, \((X, Y)\), where \(X \in [0,1]\) is the proportion of the time until the midterm that you actually study for it, and \(Y \in [0,1]\) is your percentage score on the exam. Set up but **DO NOT EVALUATE** any of your answers. Take care in setting up the limits of integration. The joint PDF is:

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\[f. \text{ Are } X \text{ and } Y \text{ independent?}\]
**Joint PDFs (Example 2)**

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\]

**f. Are \(X\) and \(Y\) independent?**

No, \(\Omega_{X,Y} \neq \Omega_X \times \Omega_Y\) (not a rectangle).
5.3 Law of Total Expectation
Agenda

- Conditional Expectation
- Law of Total Expectation (LTE)
**Conditional Expectation**

**Conditional Expectation**: Let $X$ be a discrete random variable. Then, the conditional expectation of $X$ given $A$ is

$$
E[X \mid A] = \sum_{x \in \Omega_X} x \cdot P(X = x \mid A)
$$

Linearity of expectation still applies to conditional expectation: $E[X + Y \mid A] = E[X \mid A] + E[Y \mid A]$
Law of Total Expectation

Law of Total Expectation (Event Version): Let $X$ be a random variable, and let events $A_1, ..., A_n$ partition the sample space. Then,

$$E[X] = \sum_{i=1}^{n} E[X \mid A_i] P(A_i)$$
Law of Total Expectation: Application

System that fails in step $i$ independently with probability $p$

$X \ # \ steps \ to \ fail$

$E(X) \ ?$

Let $A$ be the event that system fails in first step.
Law of Total Expectation: Application

System that fails in step \(i\) independently with probability \(p\)

\(X\) # steps to fail

\(E(X)\)?

Let \(A\) be the event that system fails in first step.

\[
E(X) = E(X|A)Pr(A) + E(X|\overline{A})Pr(\overline{A})
\]

\[
= p + (1 + E(X))(1 - p)
\]

\[
= 1 + (1 - p)E(X)
\]

\[
E(X) = \frac{1}{p}
\]
**Linearity of expectation applies**

To conditional expectation too!!

\[
E(X + Y \mid A) = E(X \mid A) + E(Y \mid A)
\]

\[
E(aX + b \mid A) = a \cdot E(X \mid A) + b
\]
Law of total Expectation (RV version)

Law of Total Expectation (RV Version): Suppose $X$ and $Y$ be discrete random variables. Then,

$$
E[X] = \sum_y E[X \mid Y = y] p_Y(y)
$$
**Problem**

The number of people who enter an elevator on the ground floor is a Poisson random variable with mean 10. If there are N floors above the ground floor, and if each person is equally likely to get off at any one of the N floors, independently of where the others get off, compute the expected number of stops that the elevator will make before discharging all the passengers.