5.2 Joint Continuous Distributions

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Most slides by Alex Tsun

$$f_{XY}(x,y)\,dx\,dy \approx \Pr(x \le X \le x+dx,\ y \le Y \le y+dy)$$

RECAP

$\Pr(X \le x, Y \le y)$

	Discrete	Continuous
Joint PMF/PDF	$p_{X,Y}(x,y) = \mathbb{P}(X=x, Y=y)$	$f_{X,Y}(x,y) \neq \mathbb{P}(X=x, Y=y)$
Joint range/support $\Omega_{X,Y}$	$\{(x,y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x,y) > 0\}$	$\{(x,y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x,y) > 0\}$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t \le x, s \le y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(t,s)\,ds\,dt$
Normalization	$\sum_{x,y} p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$
Expectation	$\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y) p_{X,Y}(x,y)$	$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy$

Independence X, Y

$\forall x \in \Omega_X\ \forall y \in \Omega_Y$

$$p_{XY}(x,y) = P(X=x)P(Y=y) = p_X(x)p_Y(y)$$

$\forall x,y \in \Omega_X, \Omega_Y$

$$f_{XY}(x,y) = f_X(x) f_Y(y)$$

$x \in \Omega_X \quad y \in \Omega_Y$

$\Omega_{X,Y} = \Omega_X \times \Omega_Y$ necessary for independence

$\Omega_{X,Y}$

missing

$\Omega_X = \{1,2,3\}$ $\Omega_Y = \{2,4\}$ $\Omega_{X,Y} = \{(1,2),(1,4),(2,2),(3,2),(3,4)\}$

$(2,4)$

$P_{XY}(2,4) = 0$

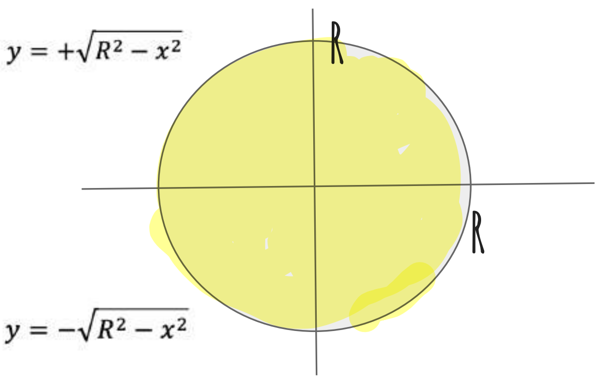
$\Omega_X = [0,1]$ $\Omega_Y = [0,2]$ $\Omega_{X,Y} = \{(x,y) \mid 0 \le x \le 1,\ x \le y \le 2\}$

Joint PDFs (Example 1)

Suppose (X,Y) are jointly and uniformly distributed on the circle of radius R centered at the origin (e.g., a dart throw). Set up but **DO NOT EVALUATE** any of your answers. Take care in setting up the limits of integration.

a. Find and sketch the joint range $\Omega_{X,Y}$.

$$\Omega_{X,Y} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le R^2\}$$



b) Find $f_{XY}(x,y)$

$$f_{XY}(x,y) = \begin{cases} c & x^2+y^2 \le R^2 \\ 0 & \text{otherwise.} \end{cases}$$

$$\iint_{x^2+y^2 \le R^2} c \, dx\, dy = 1$$

$$= c \iint_{x^2+y^2 \le R^2} dx\, dy = c\pi R^2 = 1$$

$$\Rightarrow c = \frac{1}{\pi R^2}$$

c) Find $f_X(x)$ and Ω_X

$$f_{XY} = \begin{cases} \frac{1}{\pi R^2} & x^2+y^2 \le R^2 \\ 0 & \text{o.w.} \end{cases}$$

$\Omega_X = ?$

a) $[-\sqrt{R^2-x^2}, \sqrt{R^2+x^2}]$

b) $[-R, R]$

c) $[0, 1]$

d) I dont know

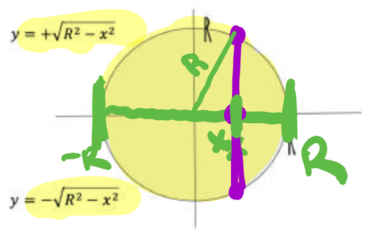
$$f_X(x) = \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \underbrace{f_{XY}(x,y)}_{\frac{1}{\pi R^2}} \, dy$$

$$= \begin{cases} \frac{2\sqrt{R^2-x^2}}{\pi R^2} & x^2 \le R^2 \\ 0 & \text{o.w.} \end{cases}$$

$x^2 + h^2 = R^2$

$h = \sqrt{R^2 - x^2}$

$\Omega_{X,Y} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le R^2\}$



e) Are X & Y independent?

a) yes

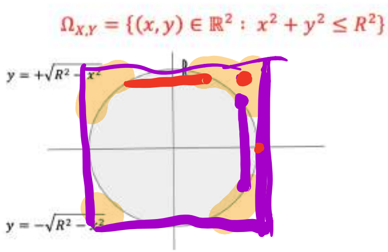
b) no

$\forall x \in \Omega_X \quad \forall y \in \Omega_Y$

$$f_{XY}(x,y) = f_X(x)\, f_Y(y)$$

$x = 0.99R \quad y = 0.99R$

$x^2 + y^2 > R^2$



$\{(x,y) \mid \underbrace{-R \le x \le R}_{\Omega_X},\ \underbrace{-R \le y \le R}_{\Omega_Y}\}$

if $\Omega_{XY} \neq \Omega_X \times \Omega_Y$

then X Y not indep.

$$\Omega_{XY} = \Omega_X \times \Omega_Y$$

necessary for indep but not sufficient

Another example

$\Omega_X = \{0,1\}$ $\Omega_Y = \{0,1\}$

$\Omega_{XY} = \{(0,0)\ (0,1)\ (1,0)\ (1,1)\}$

$$P_{X,Y}(x,y) = \begin{cases} \frac{1}{3} & (x,y) = \{(0,0),(1,1)\} \\ \frac{1}{6} & (x,y) = \{(0,1),(1,0)\} \end{cases}$$

X \ Y	0	1
0	$\frac{1}{3}$	$\frac{1}{6}$
1	$\frac{1}{6}$	$\frac{1}{3}$

$$P_X(x) = \begin{cases} \frac{1}{2} & x=0 \\ \frac{1}{2} & x=1 \end{cases} \qquad P_Y(y) = \begin{cases} \frac{1}{2} & x=0 \\ \frac{1}{2} & x=1 \end{cases}$$

$$P_X(0) \cdot P_Y(0) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \neq P_{XY}(0,0) = \frac{1}{3}$$

In this example $\Omega_{X,Y} = \Omega_X \times \Omega_Y$

but X & Y are not independent

Joint PDFs (Example 1)

Suppose (X, Y) are jointly and uniformly distributed on the circle of radius R centered at the origin (e.g., a dart throw). Set up but **DO NOT EVALUATE** any of your answers. Take care in setting up the limits of integration.

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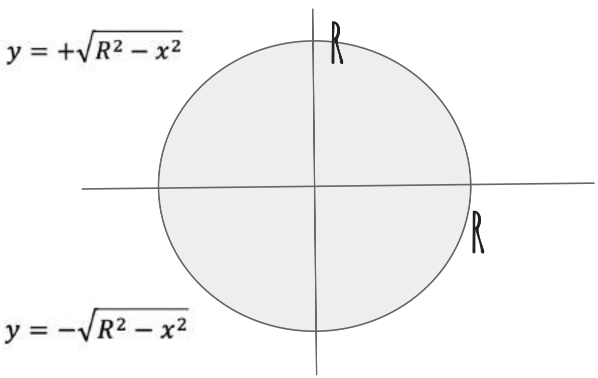
$$\Omega_{X,Y} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq R^2\}$$



b. Write an expression for the joint PDF $f_{X,Y}(x, y)$ and carefully define it for all $x, y \in \mathbb{R}$.

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\pi R^2}, & x, y \in \Omega_{X,Y} \\ 0, & \text{otherwise} \end{cases}$$

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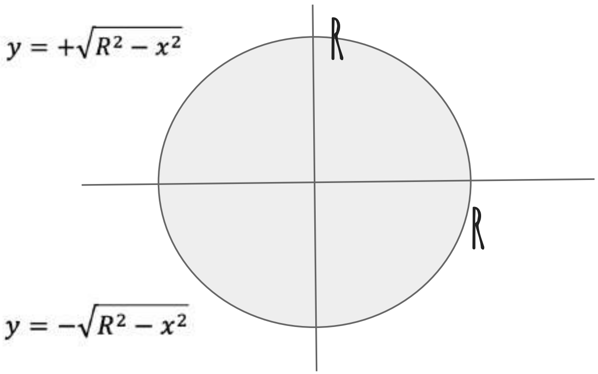

c. Find Ω_X and write an expression that we could evaluate to find $f_X(x)$.

$$\Omega_X = [-R, R]$$

$$f_X(x) = \int_{-\sqrt{R^2-x^2}}^{+\sqrt{R^2-x^2}} f_{X,Y}(x, y)dy$$

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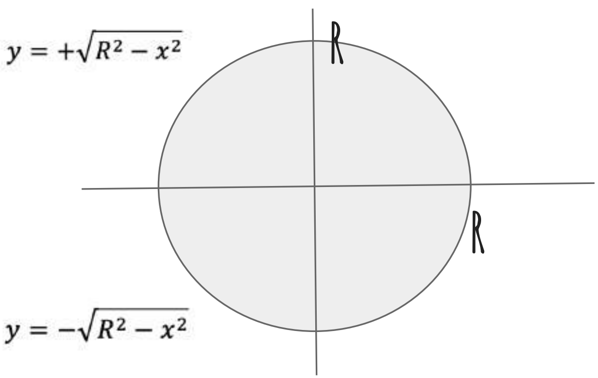

d. Let Z the distance from the center that the dart falls. Find Ω_Z and write an expression for $E[Z]$.

$$Z = \sqrt{X^2 + Y^2}$$

$$\Omega_Z = [0, R]$$

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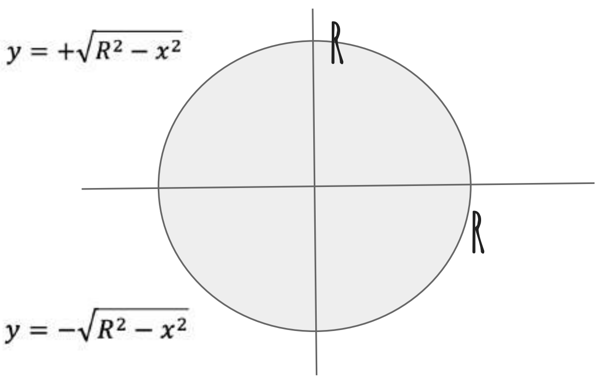

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$$Z = \sqrt{X^2 + Y^2}$$

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$$E[Z] = \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{+\sqrt{R^2-x^2}} \sqrt{x^2 + y^2} f_{X,Y}(x, y) dy dx$$

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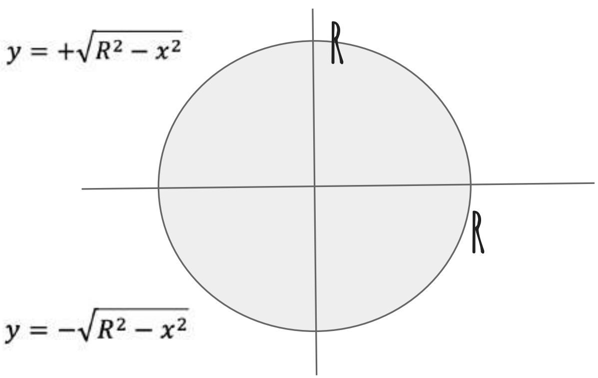

e. Are X and Y independent?

No, $\Omega_{X,Y} \neq \Omega_X \times \Omega_Y$ (not a rectangle).

$$0 = f_{X,Y}(0.99R, 0.99R) \neq f_X(0.99R) f_Y(0.99R) > 0$$

Random Picture

Joint PDFs (Example 2)

Consider a continuous joint distribution, (X, Y), where $X \in [0,1]$ is the proportion of the time until the midterm that you actually study for it, and $Y \in [0,1]$ is your percentage score on the exam. Set up but **DO NOT EVALUATE** any of your answers. Take care in setting up the limits of integration. The joint PDF is:

$$f_{X,Y}(x,y) = \begin{cases} ce^{-(y-x)}, & x, y \in [0,1] \text{ and } y \geq x \\ 0, & y < x \end{cases}$$

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a. Sketch the joint range $\Omega_{X,Y}$, and interpret it in English.

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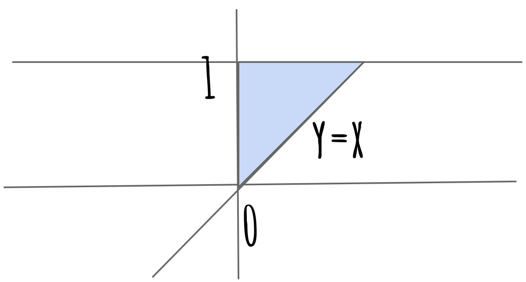
a. Sketch the joint range $\Omega_{X,Y}$, and interpret it in English.

Your score is at least the proportion of the time you study.



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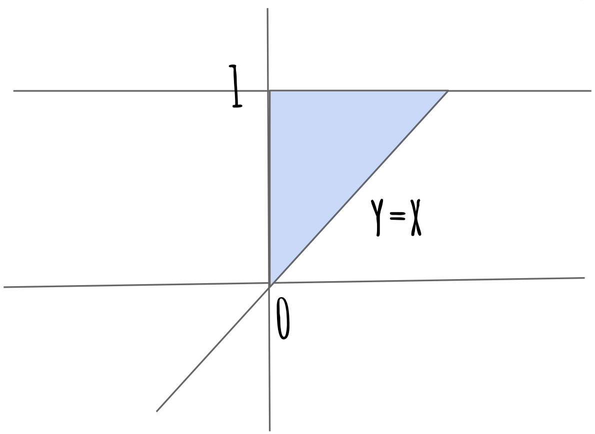

b. Write an expression that we could evaluate to find c.

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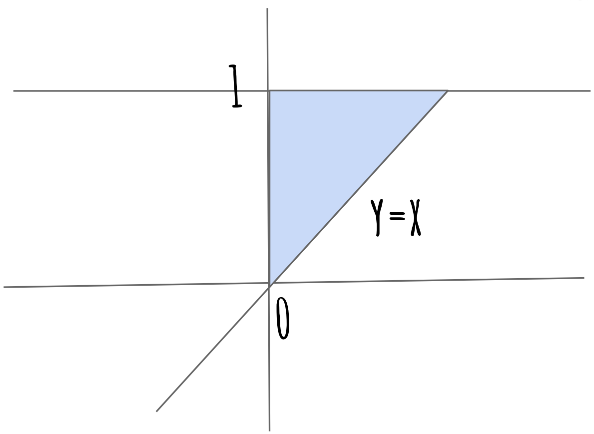

b. Write an expression that we could evaluate to find c.

$$c = \frac{1}{\int_0^1 \int_x^1 e^{-(y-x)} dy\, dx} = \frac{1}{\int_0^1 \int_0^y e^{-(y-x)} dx\, dy}$$

Joint PDFs (Example 2)

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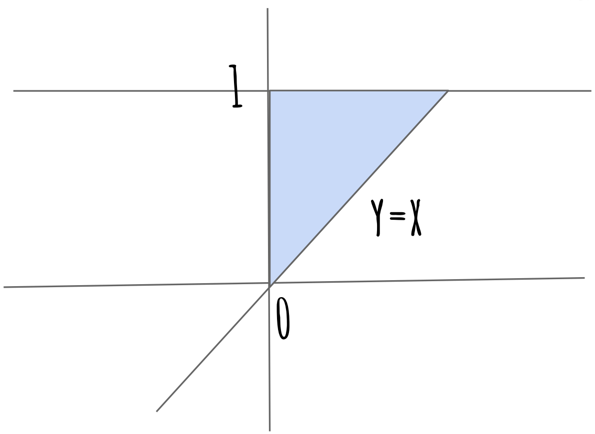

c. Find Ω_Y and write an expression that we could evaluate to find $f_Y(y)$.

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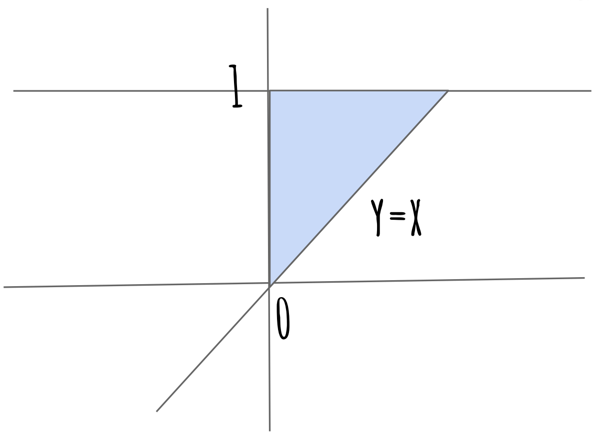

c. Find Ω_Y and write an expression that we could evaluate to find $f_Y(y)$.

$$\Omega_Y = [0,1]$$

$$f_Y(y) = \int_0^y ce^{-(y-x)}\,dx$$

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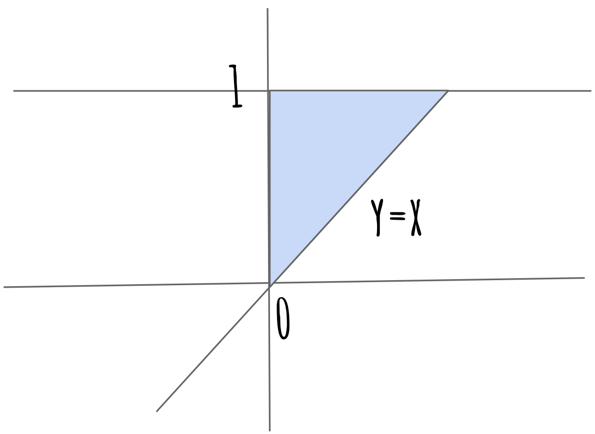

d. Write an expression that we could evaluate to find $P(Y \geq 0.9)$.

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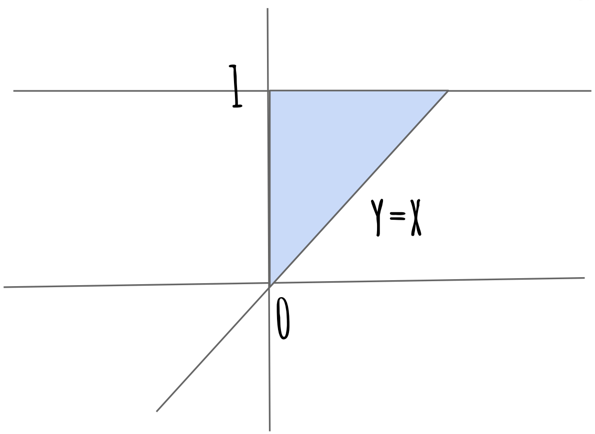

d. Write an expression that we could evaluate to find $P(Y \geq 0.9)$.

$$P(Y \geq 0.9) = \int_{0.9}^{1} f_Y(y)dy = \int_{0.9}^{1}\int_{0}^{y} ce^{-(y-x)}dx\,dy$$

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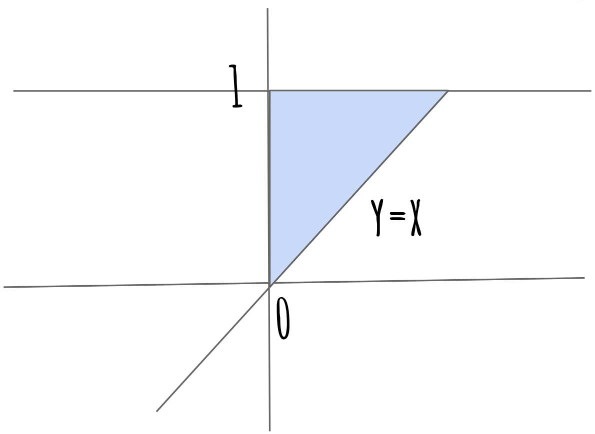

e. Write an expression that we could evaluate to find $E[Y]$.

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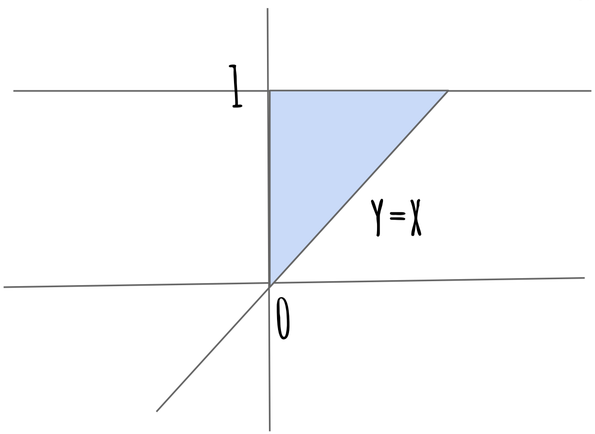

e. Write an expression that we could evaluate to find $E[Y]$.

$$E[Y] = \int_0^1 y f_Y(y) dy = \int_0^1 \int_0^y cye^{-(y-x)} dx\, dy$$

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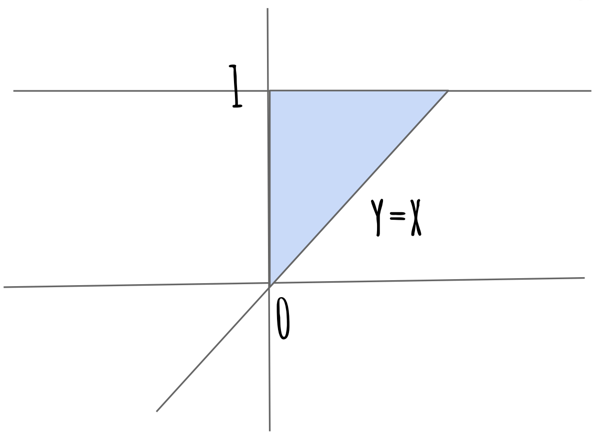

f. Are X and Y independent?

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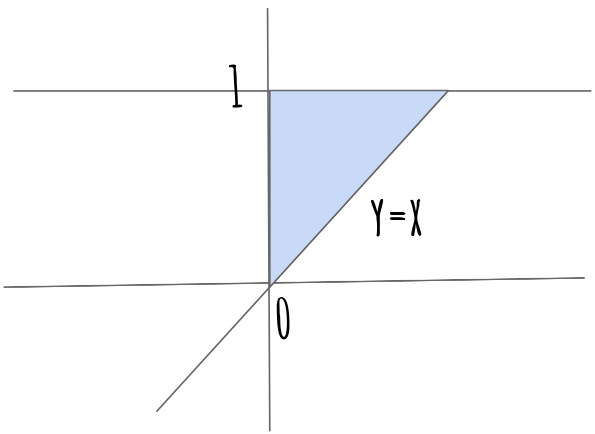


f. Are X and Y independent?

No, $\Omega_{X,Y} \neq \Omega_X \times \Omega_Y$ (not a rectangle).

5.3 Law of Total Expectation

Agenda

- Conditional Expectation
- Law of Total Expectation (LTE)

Conditional Expectation

Conditional Expectation: Let X be a discrete random variable. Then, the conditional expectation of X given A is

$$\mathbb{E}[X \mid A] = \sum_{x \in \Omega_X} x\mathbb{P}(X = x|A)$$

Linearity of expectation still applies to conditional expectation: $\mathbb{E}[X + Y \mid A] = \mathbb{E}[X \mid A] + \mathbb{E}[Y \mid A]$

$X \sim \text{Poisson}(\lambda)$

$\Rightarrow A: X \leq 1$

$$E(X|A) = \sum_{i=0}^{\infty} i \Pr(X=i \mid X \leq 1)$$

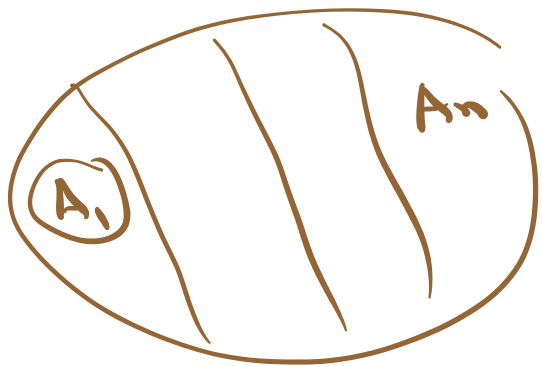
$$= 0 \cdot \Pr(X=0 \mid X \leq 1) + 1 \cdot \Pr(X=1 \mid X \leq 1) + \ldots 0$$

Law of Total Expectation

Law of Total Expectation (Event Version): Let X be a random variable, and let events $A_1, \ldots, A_n$ partition the sample space. Then,

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid A_i]\, \mathbb{P}(A_i)$$

follows directly
Law of Total Prob.



$$E(X) = E(X \mid A)P(A) + E(X \mid \bar{A})P(\bar{A})$$

Law of Total Expectation : Application

System that fails in step i independently with probability p

X # steps to fail

E(X) ?

Let A be the event that system fails in first step.

$p,\ 1-p,\ p, \ldots$

Geo(p)

$E(X) = \frac{1}{p}$

$E(X \mid \bar{A}) = ?$

a) $E(X)$
b) 0
c) 1
d) $1 + E(X)$

first step. F p. S 1-p.

x	outcomes
1	F
2	SF
3	SSF
4	SSSF

$E(X) = E(X \mid A)\Pr(A) + E(X \mid \bar{A})\Pr(\bar{A})$

(fails in first step) (success in first step.)

$= 1 \cdot p + E(X \mid \text{success})(1-p)$, where $E(X \mid \text{success}) = 1 + E(X)$

$E(X \mid \text{1st step is success}) = 1 + E(X)$

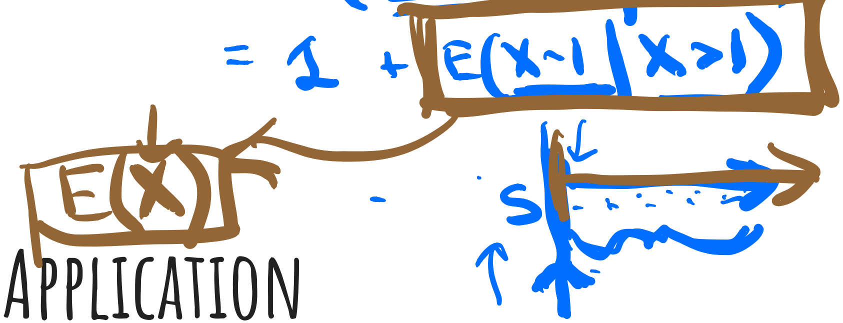
$= E(1 + X - 1 \mid X > 1)$

$E(X) = p + (1 + E(X))(1-p)$

$= p + 1 - p + E(X)(1-p)$

$p E(X) = 1 \Rightarrow E(X) = \frac{1}{p}$

Law of Total Expectation : Application



System that fails in step i independently
with probability p
X # steps to fail
E(X) ?

Let A be the event that system fails in first step.

$$E(X) = E(X|A)Pr(A) + E(X|\overline{A})Pr(\overline{A})$$

$$= p + (1 + E(X))(1-p)$$

$$= 1 + (1-p)E(X)$$

$$E(X) = \frac{1}{p}$$

Problem

The number of people who enter an elevator on the ground floor is a Poisson random variable with mean 10. If there are N floors above the ground floor, and if each person is equally likely to get off at any one of the N floors, independently of where the others get off, compute the expected number of stops that the elevator will make before discharging all the passengers.

$$Y_i = \begin{cases} 1 & \text{elevator stops on } i^{th} \text{ floor} \\ 0 & \text{otherwise} \end{cases}$$

$E(Y_i) = ?$

a) $\frac{20}{N}$

b) $1 - \left(1 - \frac{1}{N}\right)^{20}$

c) 1

d) I don't know