Joint Distributions

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Most slides by Alex Tsun

Joint distributions

- Given all of its user's ratings for different movies and any preferences you have expressed, Netflix wants to recommend a new movie for you.
- Given a bunch of medical data correlating symptoms and personal history with diseases, predict what is ailing a person with a particular medical history and set of symptoms.
- Given current traffic, pedestrian locations, weather, lights, etc, decide whether self-driving car should slow down or come to a stop?

5.1 Joint Discrete Distributions

Agenda

- Cartesian Products of Sets
- Joint PMFs and Expectation
- Marginal PMFs

Cartesian Product of Sets

Cartesian Product: Let A, B be sets. The Cartesian product of A and B is denoted

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

A small example:

$$\{1,2,3\} \times \{4,5\} = \{(1,4), (1,5), (2,4), (2,5), (3,4), (3,5)\}$$

Another example: The xy-plane (2D space) is denoted

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R})\}$$

If A, B are finite sets, then $|A \times B| = |A| \cdot |B|$ by the product rule of counting.

Example: ratings

Let X be the value of the blue die, and Y the value of the red die. Specify

$\Omega_X = \{1,2,3,4\}$ $\Omega_Y = \{1,2,3,4\}$

$\Omega_{X,Y} = \Omega_X \times \Omega_Y$

Specify the joint PMF $p_{X,Y}(x,y) = P(X = x, Y = y)$ for $x, y \in \Omega_{X,Y}$.

X\Y	1	2	3	4
1				
2				
3				
4				

Example: weird dice Again

Suppose I roll two fair 4-sided die independently. Let X be the value of the blue die, and Y the value of the red die. Specify

$\Omega_X = \{1,2,3,4\}$ $\Omega_Y = \{1,2,3,4\}$

$$\Omega_{X,Y} = \Omega_X \times \Omega_Y$$

Specify the joint PMF $p_{X,Y}(x,y) = P(X = x, Y = y)$ for $x, y \in \Omega_{X,Y}$.

$$p_{X,Y}(x,y) = \begin{cases} 1/16, & x,y \in \Omega_{X,Y} \\ 0, & \text{otherwise} \end{cases}$$

X\Y	1	2	3	4
1	1/16	1/16	1/16	1/16
2	1/16	1/16	1/16	1/16
3	1/16	1/16	1/16	1/16
4	1/16	1/16	1/16	1/16

Joint PMFs and Expectation

Joint PMFs: Let X, Y be discrete random variables. The joint PMF of X and Y is

$$p_{X,Y}(a,b) = P(X = a, Y = b)$$

The joint range is

$$\Omega_{X,Y} = \{(c,d): p_{X,Y}(c,d) > 0\} \subseteq \Omega_X \times \Omega_Y$$

Note that

$$\sum_{(s,t)\in\Omega_{X,Y}} p_{X,Y}(s,t) = 1$$

Example: weird dice Again

Suppose I roll two fair 4-sided die independently. Let X be the value of the blue die, and Y the value of the red die. Let $U = \min\{X, Y\}$ and $V = \max\{X, Y\}$.

$\Omega_U = \{1,2,3,4\}$ $\Omega_V = \{1,2,3,4\}$

$\Omega_{U,V} = \{(u, v) \in \Omega_U \times \Omega_V : u \leq v\} \neq \Omega_U \times \Omega_V$

Specify the joint PMF $p_{U,V}(u, v) = P(U = u, V = v)$ for $u, v \in \Omega_{U,V}$.

U\V	1	2	3	4
1				
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Specify the joint PMF $p_{U,V}(u, v) = P(U = u, V = v)$ for $u, v \in \Omega_{U,V}$.

$$p_{U,V}(u,v) = \begin{cases} 2/16, & u, v \in \Omega_U \times \Omega_V, \quad v > u \\ 1/16, & u, v \in \Omega_U \times \Omega_V, \quad v = u \\ 0, & \text{otherwise} \end{cases}$$

U\V	1	2	3	4
1	1/16	2/16	2/16	2/16
2	0	1/16	2/16	2/16
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Example: weird dice Again

Suppose I roll two fair 4-sided die independently. Let X be the value of the blue die, and Y the value of the red die. Let $U = \min\{X, Y\}$ and $V = \max\{X, Y\}$.

What is $p_U(u)$ for $u \in \Omega_U$?

$$p_U(u) = \begin{cases} & u = 1 \\ & u = 2 \\ & u = 3 \\ & u = 4 \end{cases}$$

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What is $p_U(u)$ for $u \in \Omega_U$?

$$p_U(u) = \begin{cases} 7/16, & u = 1 \\ 5/16, & u = 2 \\ 3/16, & u = 3 \\ 1/16, & u = 4 \end{cases}$$

U\V	1	2	3	4
1	1/16	2/16	2/16	2/16
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Joint PMFs and Expectation

Joint PMFs: Let X, Y be discrete random variables. The joint PMF of X and Y is

$$p_{X,Y}(a,b) = P(X = a, Y = b)$$

The joint range is

$$\Omega_{X,Y} = \{(c,d): p_{X,Y}(c,d) > 0\} \subseteq \Omega_X \times \Omega_Y$$

Note that

$$\sum_{(s,t)\in\Omega_{X,Y}} p_{X,Y}(s,t) = 1$$

If $g: \mathbb{R}^2 \to \mathbb{R}$ is a function, then

$$E[g(X,Y)] = \sum_x \sum_y g(x,y) p_{X,Y}(x,y)$$

.

Marginal PMFs

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$$p_Y(d) = \sum_{c \in \Omega_X} p_{X,Y}(c, d)$$

(Extension) If Z is also a discrete random variable, then the marginal PMF of Z is

$$p_Z(z) = \sum_{x \in \Omega_X} \sum_{y \in \Omega_Y} p_{X,Y,Z}(x, y, z)$$

Independence

Independence (DRVs): Discrete random variables X, Y are independent, written $X \perp Y$, if for all $x \in \Omega_X$ and $y \in \Omega_Y$,

$$p_{X,Y}(x, y) = p_X(x)p_Y(y)$$

Recall $\Omega_{X,Y} = \{(x, y): p_{X,Y}(x, y) > 0\} \subseteq \Omega_X \times \Omega_Y$. A necessary but not sufficient condition for independence is that $\Omega_{X,Y} = \Omega_X \times \Omega_Y$. That is, if $\Omega_{X,Y} \neq \Omega_X \times \Omega_Y$, then X and Y cannot be independent, but if $\Omega_{X,Y} = \Omega_X \times \Omega_Y$, then we have to check the condition.

This is because if there is some $(a, b) \in \Omega_X \times \Omega_Y$ but not in $\Omega_{X,Y}$, then $p_{X,Y}(a, b) = 0$ but $p_X(a) > 0$ and $p_Y(b) > 0$, violating independence.

Variance adds for Independent RVs

If X, Y are independent random variables $X \perp Y$, then

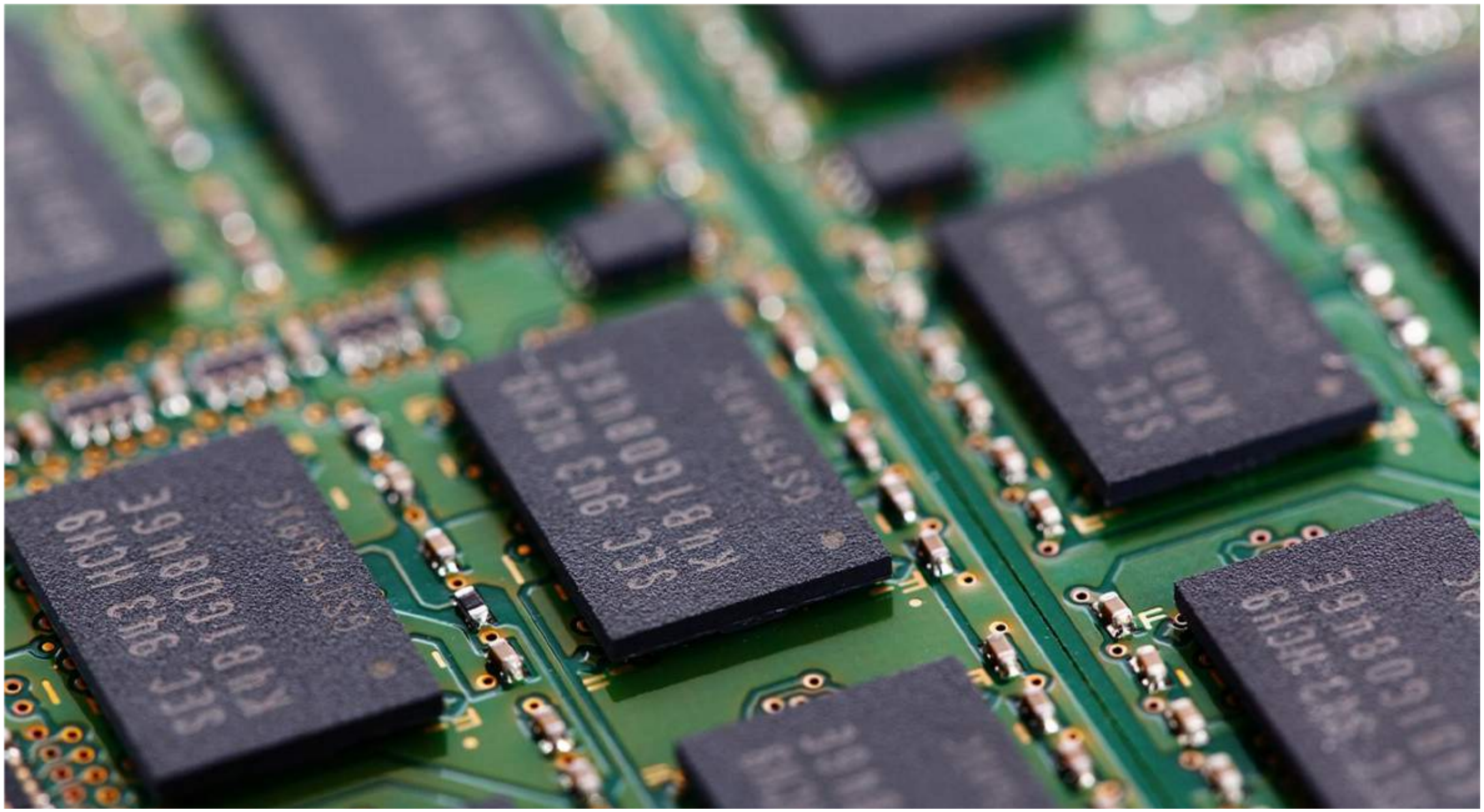
$$Var(X + Y) = Var(X) + Var(Y)$$

This property relies on the fact that they are independent, whereas linearity of expectation **always** holds, regardless. If $a, b, c \in \mathbb{R}$ are scalars, then

$$Var(aX + bY + c) = a^2Var(X) + b^2Var(Y)$$

If $X \perp Y$, then $E[XY] = E[X]E[Y]$.

Random Picture



5.2 Joint Continuous Distributions

Agenda

- Joint PDFs and Expectation
- Marginal PDFs
- Independence
- Multivariate: From Discrete to Continuous

Joint PDFs and Expectation

Joint PDFs: Let X, Y be continuous random variables. The joint PDF of X and Y is

$$f_{X,Y}(a, b)$$

The joint range is

$$\Omega_{X,Y} = \{(c, d): f_{X,Y}(c, d) > 0\} \subseteq \Omega_X \times \Omega_Y$$

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Note that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(u, v) du dv = 1$$

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Note that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(u,v)dudv = 1$$

If $g: \mathbb{R}^2 \to \mathbb{R}$ is a function, then

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(s,t)f_{X,Y}(s,t)dsdt$$

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Joint PDFs (Example 1)

Suppose (X, Y) are jointly and uniformly distributed on the circle of radius R centered at the origin (e.g., a dart throw). Set up but **DO NOT EVALUATE** any of your answers. Take care in setting up the limits of integration.

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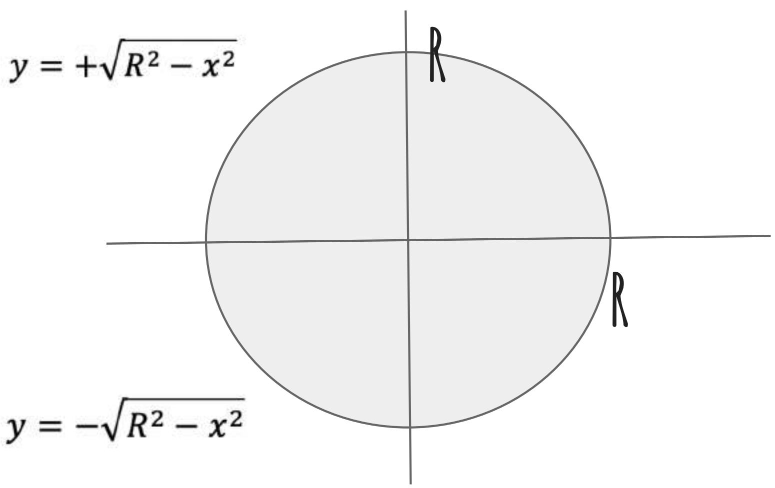

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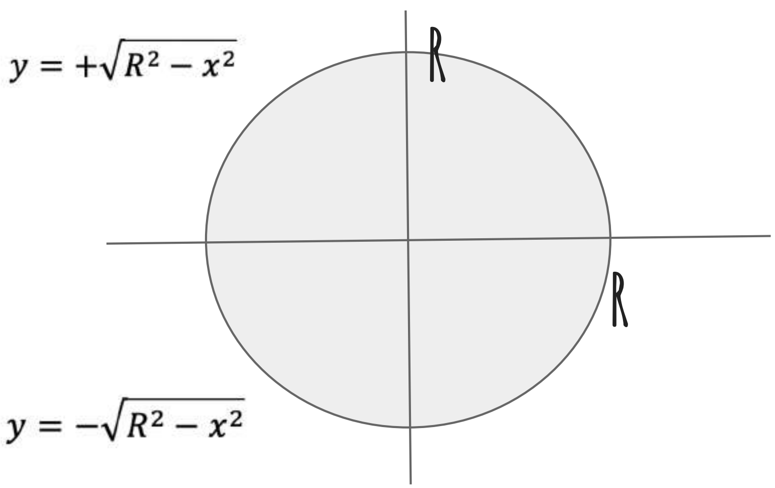

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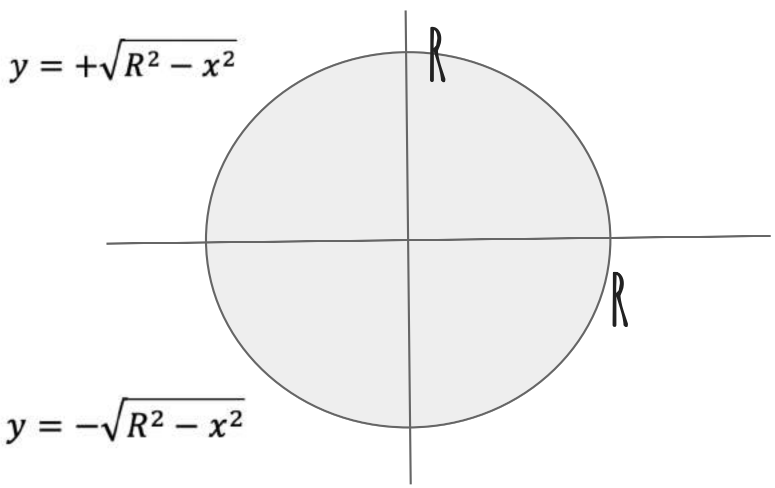

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$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\pi R^2}, & x, y \in \Omega_{X,Y} \\ 0, & \text{otherwise} \end{cases}$$

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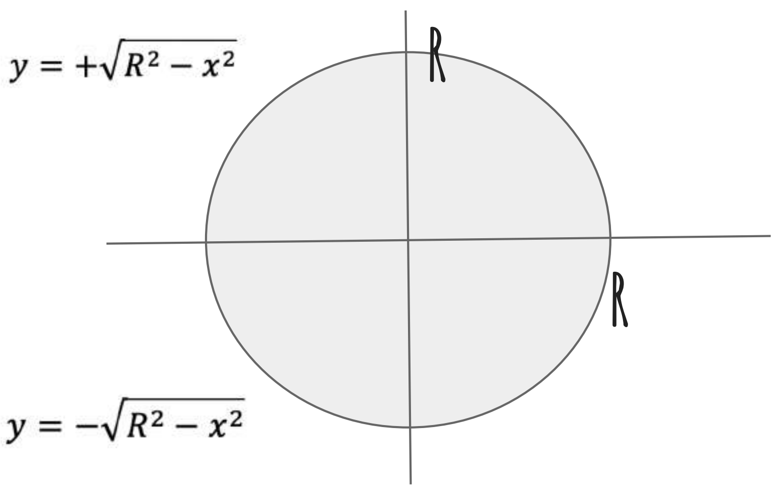

c. Find Ω_X and write an expression that we could evaluate to find $f_X(x)$.

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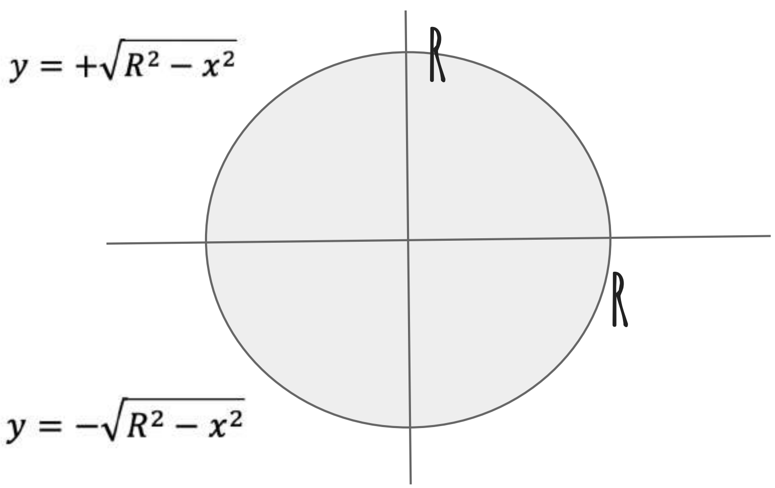

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$$\Omega_X = [-R, R]$$

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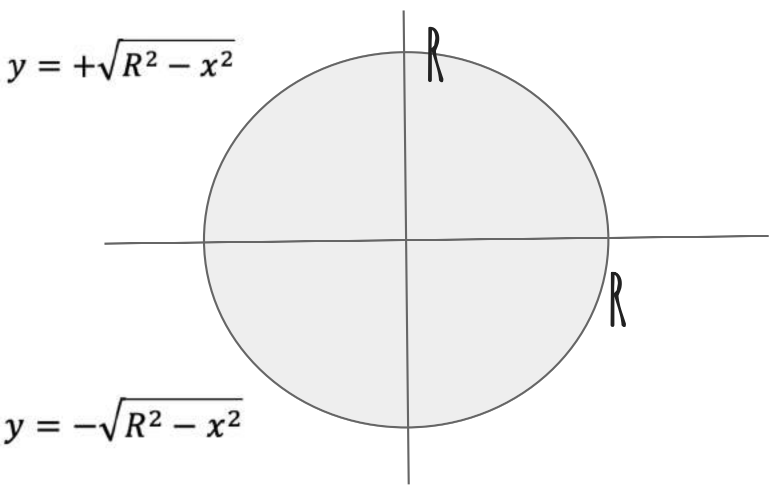

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$$\Omega_X = [-R, R]$$

$$f_X(x) = \int_{-\sqrt{R^2-x^2}}^{+\sqrt{R^2-x^2}} f_{X,Y}(x, y)\,dy$$

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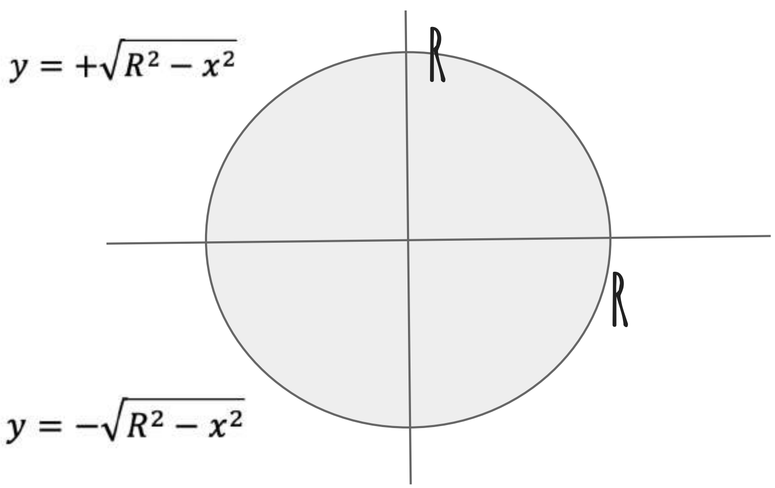

d. Let Z the distance from the center that the dart falls. Find Ω_Z and write an expression for $E[Z]$.

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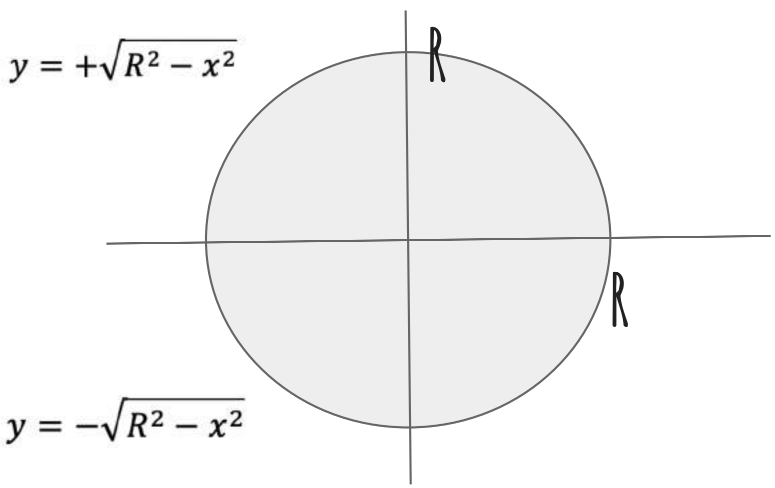

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$$\Omega_Z = [0, R]$$

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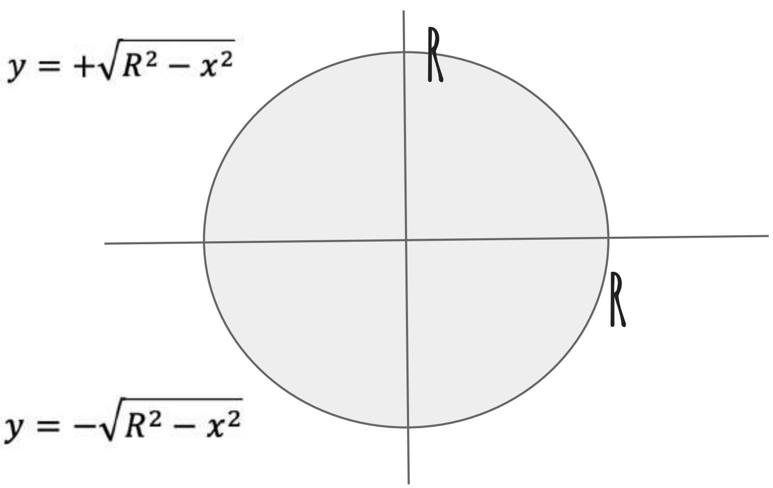

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$$Z = \sqrt{X^2 + Y^2}$$

$$\Omega_Z = [0, R]$$

$$E[Z] = \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{+\sqrt{R^2-x^2}} \sqrt{x^2 + y^2} f_{X,Y}(x, y) dy dx$$

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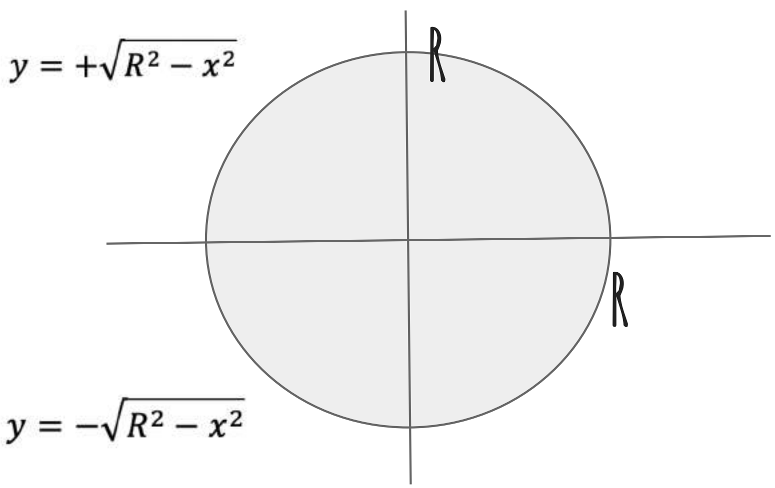

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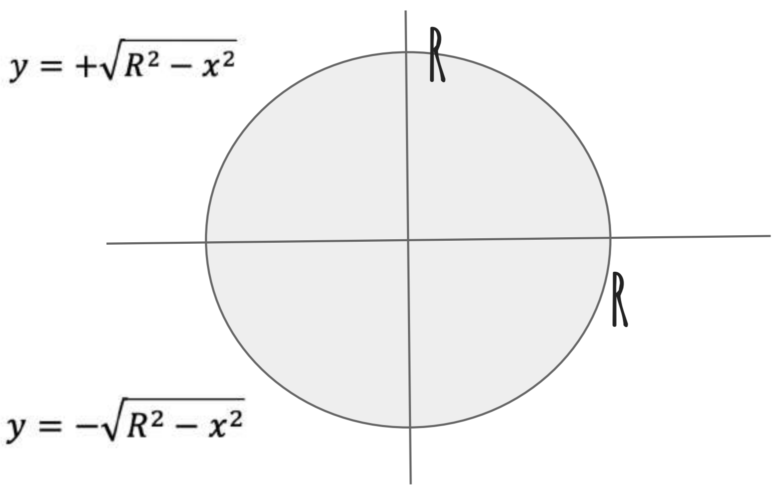

e. Are X and Y independent?

No, $\Omega_{X,Y} \neq \Omega_X \times \Omega_Y$ (not a rectangle).

$$0 = f_{X,Y}(0.99R, 0.99R) \neq f_X(0.99R) f_Y(0.99R) > 0$$

Random Picture

Joint PDFs (Example 2)

Consider a continuous joint distribution, (X, Y), where $X \in [0,1]$ is the proportion of the time until the midterm that you actually study for it, and $Y \in [0,1]$ is your percentage score on the exam. Set up but **DO NOT EVALUATE** any of your answers. Take care in setting up the limits of integration. The joint PDF is:

$$f_{X,Y}(x,y) = \begin{cases} ce^{-(y-x)}, & x, y \in [0,1] \text{ and } y \geq x \\ 0, & y < x \end{cases}$$

Joint PDFs (Example 2)

Consider a continuous joint distribution, (X,Y), where $X \in [0,1]$ is the proportion of the time until the midterm that you actually study for it, and $Y \in [0,1]$ is your percentage score on the exam. Set up but **DO NOT EVALUATE** any of your answers. Take care in setting up the limits of integration. The joint PDF is:

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a. Sketch the joint range $\Omega_{X,Y}$, and interpret it in English.

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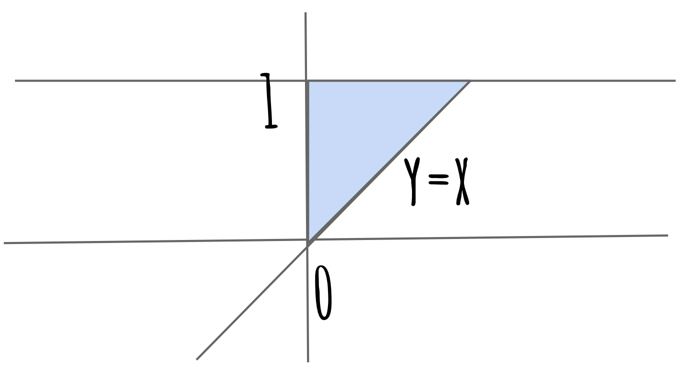
$$f_{X,Y}(x,y) = \begin{cases} ce^{-(y-x)}, & x,y \in [0,1] \text{ and } y \geq x \\ 0, & y < x \end{cases}$$

a. Sketch the joint range $\Omega_{X,Y}$, and interpret it in English.

Your score is at least the proportion of the time you study.



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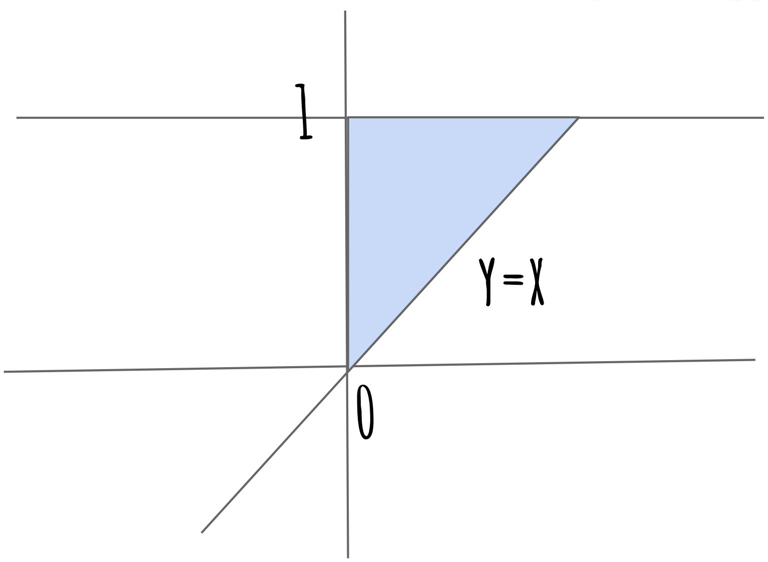

b. Write an expression that we could evaluate to find c.

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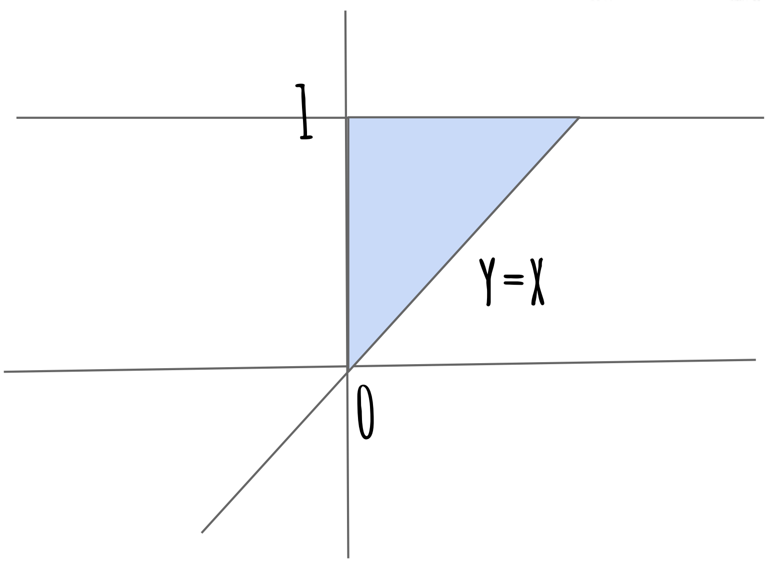

b. Write an expression that we could evaluate to find c.

$$c = \frac{1}{\int_0^1 \int_x^1 e^{-(y-x)} dy\, dx} = \frac{1}{\int_0^1 \int_0^y e^{-(y-x)} dx\, dy}$$

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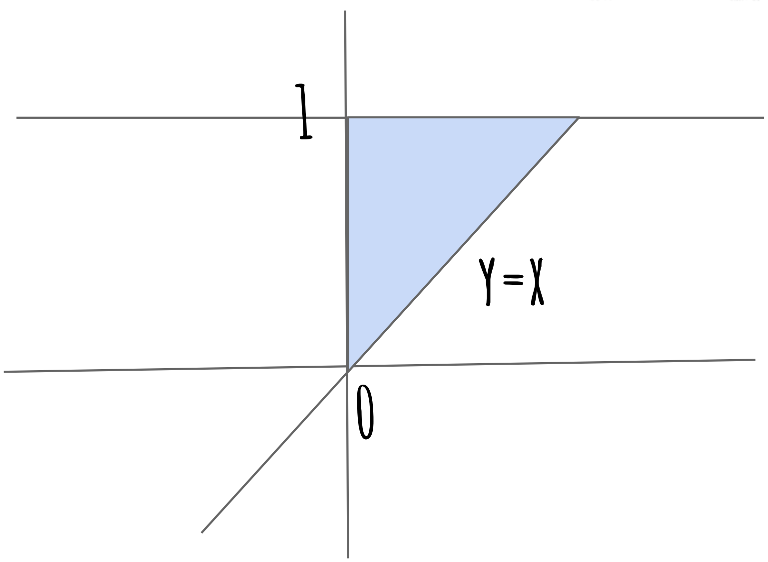

c. Find Ω_Y and write an expression that we could evaluate to find $f_Y(y)$.

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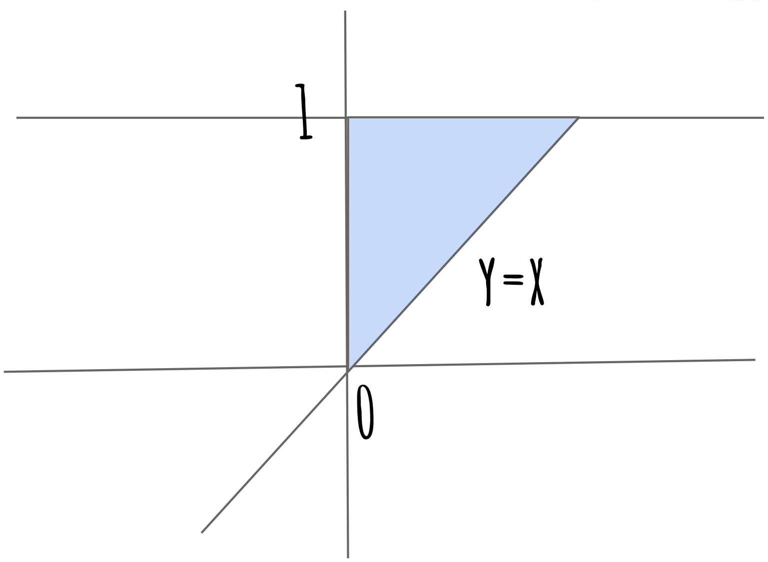

c. Find Ω_Y and write an expression that we could evaluate to find $f_Y(y)$.

$$\Omega_Y = [0,1]$$

$$f_Y(y) = \int_0^y ce^{-(y-x)}\,dx$$

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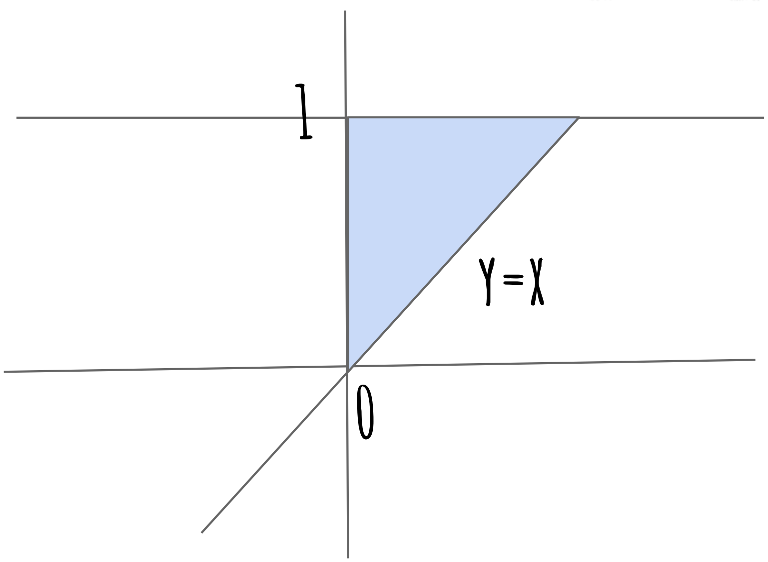

d. Write an expression that we could evaluate to find $P(Y \geq 0.9)$.

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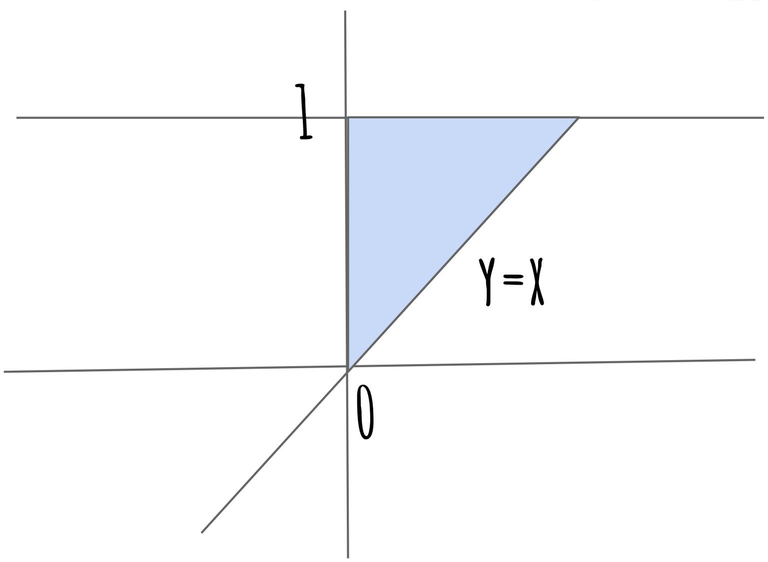

d. Write an expression that we could evaluate to find $P(Y \geq 0.9)$.

$$P(Y \geq 0.9) = \int_{0.9}^{1} f_Y(y)dy = \int_{0.9}^{1}\int_{0}^{y} ce^{-(y-x)}dx\,dy$$

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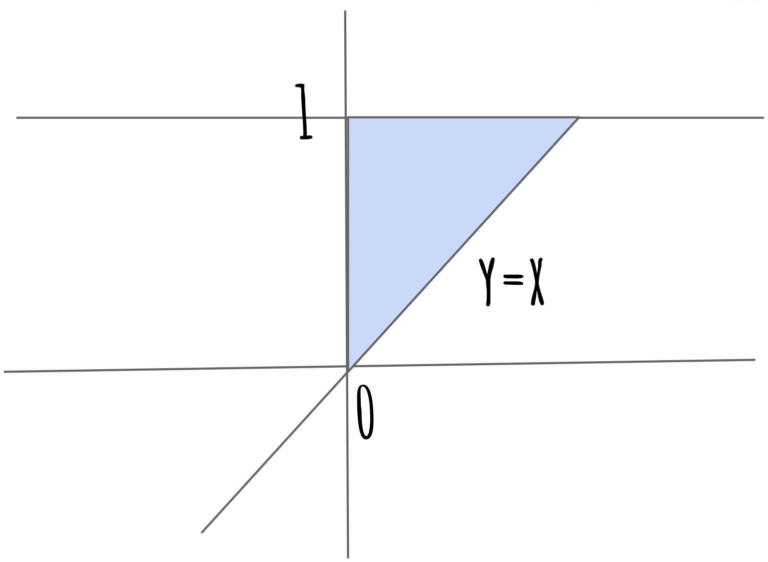

e. Write an expression that we could evaluate to find $E[Y]$.

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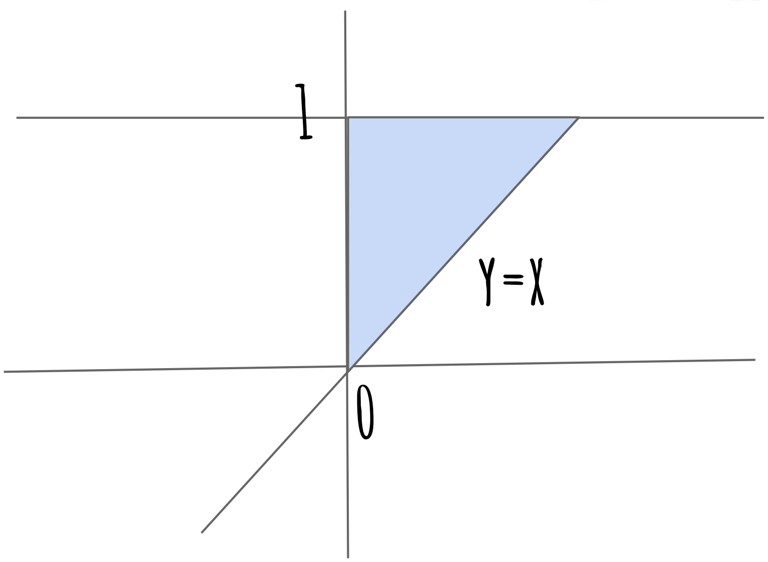

e. Write an expression that we could evaluate to find $E[Y]$.

$$E[Y] = \int_0^1 y f_Y(y) dy = \int_0^1 \int_0^y cye^{-(y-x)} dx\, dy$$

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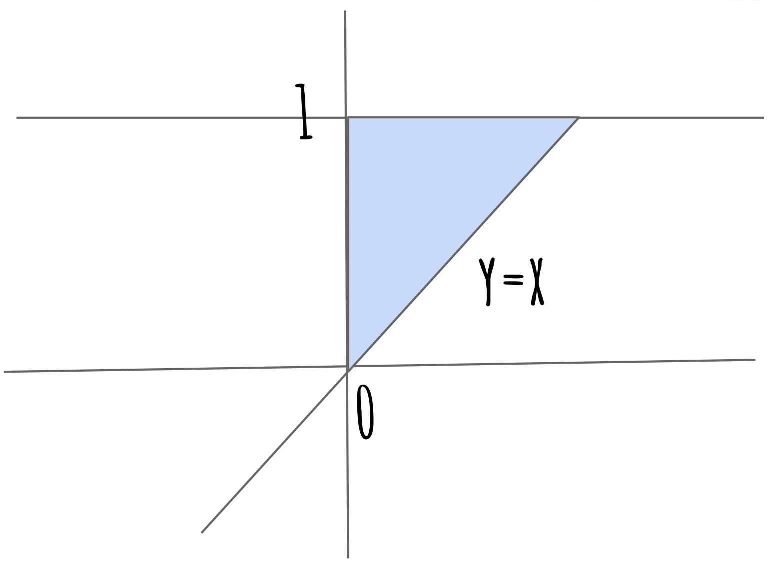

f. Are X and Y independent?

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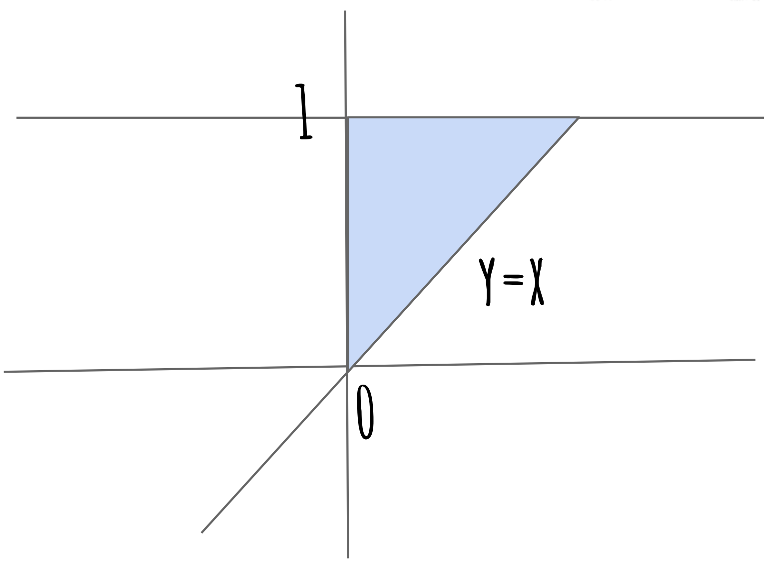

f. Are X and Y independent?

No, $\Omega_{X,Y} \neq \Omega_X \times \Omega_Y$ (not a rectangle).