# Continuous random Variables

Anna Karlin
Most Slides by Alex Tsun + Joshua Fan

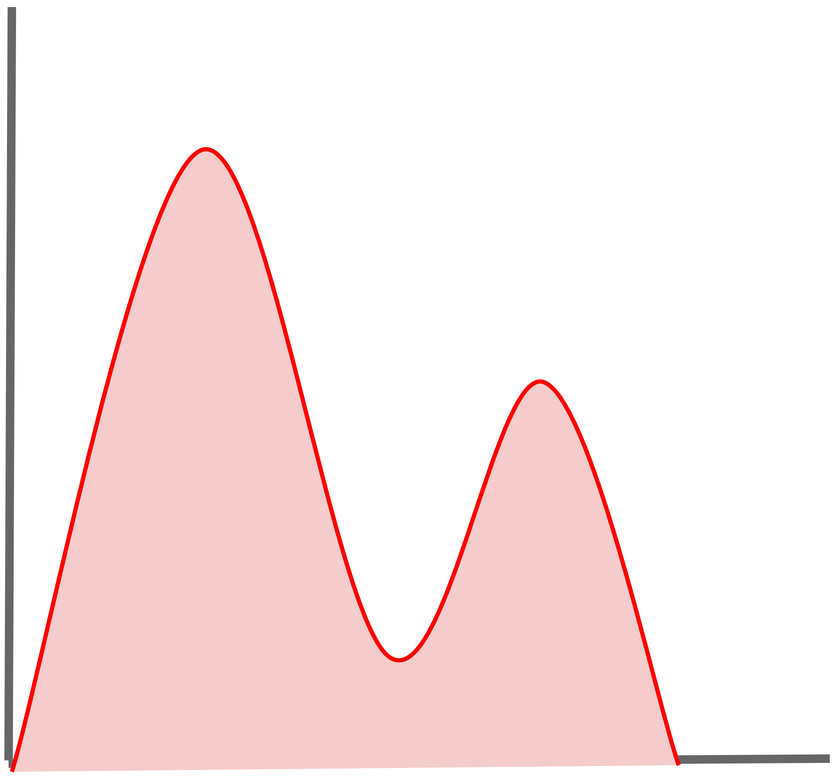
# Agenda

- Recap (PDFs and CDFs)
- The (Continuous) Uniform RV
- The Exponential RV
- Memorylessness
- The normal distribution

# From Discrete to Continuous

| | Discrete | Continuous |
|---|---|---|
| **PMF/PDF** | $p_X(x) = \mathbb{P}(X = x)$ | $f_X(x) \neq \mathbb{P}(X = x) = 0$ |
| **CDF** | $F_X(x) = \sum_{t \leq x} p_X(t)$ | $F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$ |
| **Normalization** | $\sum_x p_X(x) = 1$ | $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$ |
| **Expectation** | $\mathbb{E}[X] = \sum_x x p_X(x)$ | $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$ |
| **LOTUS** | $\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$ | $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$ |

# PDF Intuition



$$f_X(z) \geq 0 \text{ for all } z \in \mathbb{R}$$

$$\int_{-\infty}^{\infty} f_X(t)dt = 1$$

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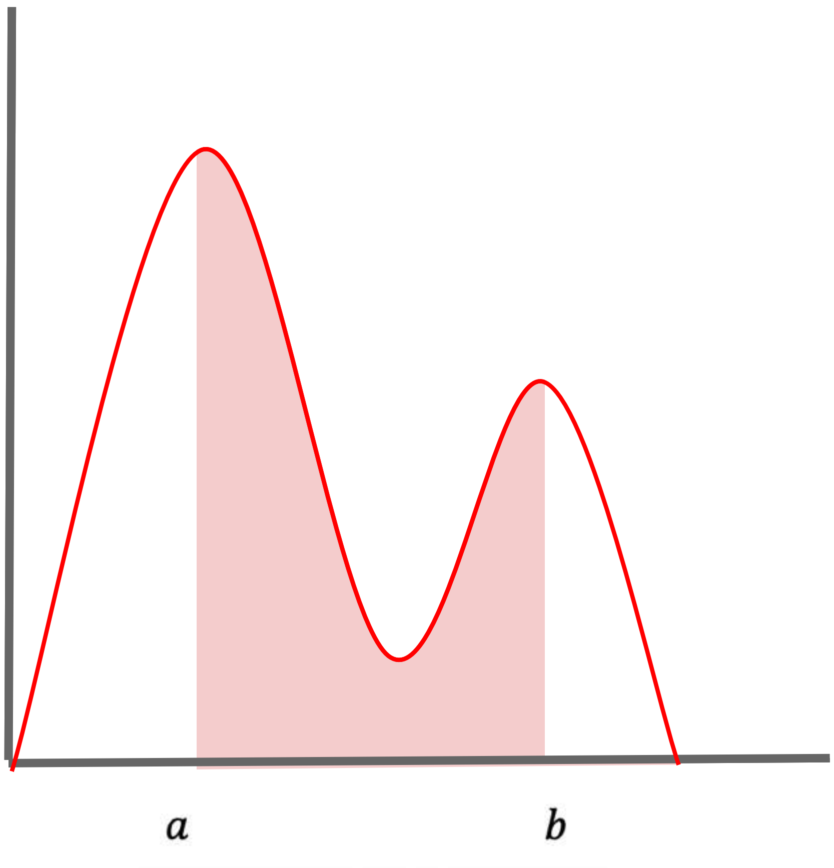


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$$P(a \leq X \leq b) = \int_a^b f_X(w)dw$$

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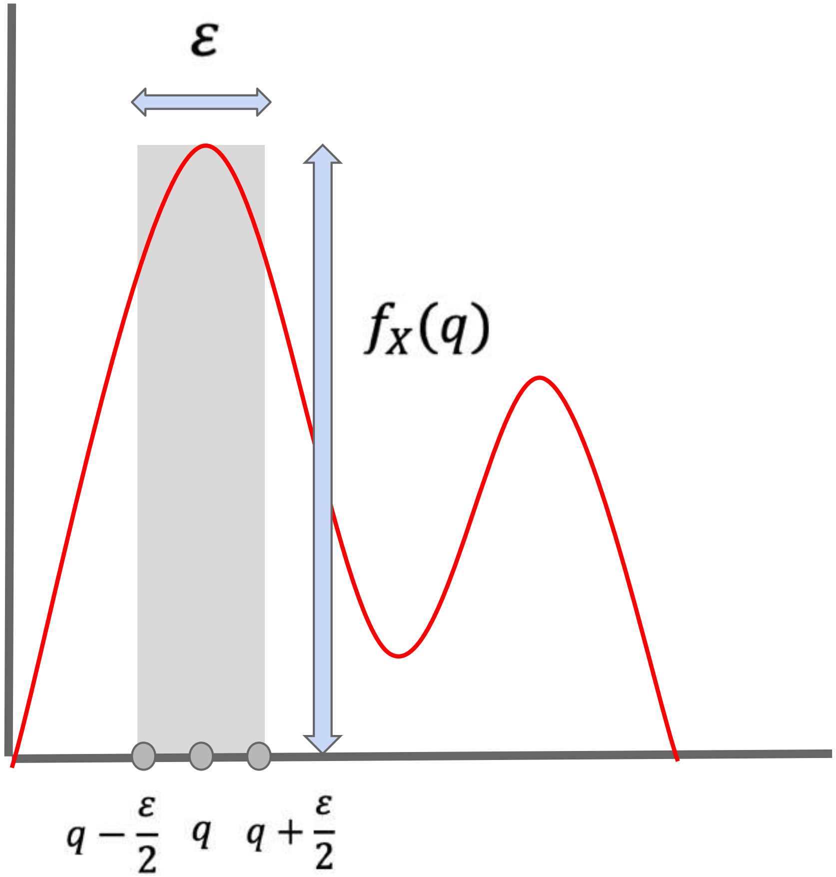


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$$P(a \leq X \leq b) = \int_a^b f_X(w)dw$$

$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(w)dw = 0$$

$$P(X \approx q) \approx P\left(q - \frac{\varepsilon}{2} \leq X \leq q + \frac{\varepsilon}{2}\right) \approx \varepsilon f_X(q)$$

# Cumulative Distribution Functions (CDFs)

**Cumulative Distribution Function (CDF):** Let $X$ be a continuous rv (one whose range is typically an interval or union of intervals). The cumulative distribution function (CDF) of $X$ is the function $F_X: \mathbb{R} \to \mathbb{R}$ such that

- $F_X(t) = P(X \leq t) = \int_{-\infty}^{t} f_X(w)dw$ for all $t \in \mathbb{R}$.
- Hence, by the Fundamental Theorem of Calculus, $\frac{d}{du}F_X(u) = f_X(u)$.
- $P(a \leq X \leq b) = F_X(b) - F_X(a)$.
- $F_X$ is monotone increasing, since $f_X \geq 0$. That is, $F_X(c) \leq F_X(d)$ for $c \leq d$.
- $\lim_{v \to -\infty} F_X(v) = P(X \leq -\infty) = 0$.
- $\lim_{v \to +\infty} F_X(v) = P(X \leq +\infty) = 1$.

# 4.2 Zoo of Continuous RVs

# The Uniform (Continuous) RV

**Uniform (Continuous) RV:** $X \sim Unif(a, b)$ where $a < b$ are real numbers, if and only if $X$ has the following pdf:

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & x \in [a, b] \\ 0, & \text{otherwise} \end{cases}$$

$X$ is equally likely to take on any value in $[a, b]$.

$$E[X] = \frac{a+b}{2} \qquad Var(X) = \frac{(b-a)^2}{12}$$

The cdf is

$$F_X(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \le x \le b \\ 1, & x > b \end{cases}$$

# The Exponential PDF/CDF

Recall the Poisson Process with parameter $\lambda > 0$ has events happening at average rate of $\lambda$ per unit of time forever. The exponential RV measures the time until the first occurrence of an event, so is a continuous RV with range $[0, \infty)$ (unlike the Poisson RV, which counts the number of occurrences in a unit of time, with range $\{0,1,2, \dots\}$.)

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Let $Y \sim Exp(\lambda)$ be the time until the first event. We'll compute $F_Y(t)$ and $f_Y(t)$.
Let $X(t) \sim Poi(\lambda t)$ be the # of events in the first $t$ units of time, for $t \geq 0$.

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$$P(Y > t) = P(\text{no events in first } t \text{ units}) = P(X(t) = 0) = e^{-\lambda t}\frac{(\lambda t)^0}{0!} = e^{-\lambda t}$$

$$F_Y(t) = P(Y \leq t) = 1 - P(Y > t) = 1 - e^{-\lambda t}$$

$$f_Y(t) = \frac{d}{dt}F_Y(t) = \lambda e^{-\lambda t}$$

# The Exponential RV Properties

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# The Exponential RV Properties

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_{0}^{\infty} x \cdot \lambda e^{-\lambda x} dx = \frac{1}{\lambda}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_{0}^{\infty} x^2 \cdot \lambda e^{-\lambda x} dx = \frac{2}{\lambda^2}$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

# The Exponential RV

**<u>Exponential RV:</u>** $X \sim Exp(\lambda)$, if and only if $X$ has the following pdf:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$X$ is the waiting time until the first occurrence of an event in a Poisson process with parameter $\lambda$.

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$$E[X] = \frac{1}{\lambda} \qquad Var(X) = \frac{1}{\lambda^2}$$

The cdf is

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

# Random Picture

# Memorylessness (Intuition)


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For example, let $s = 7, t = 2$. So $P(X > 9 \mid X > 7) = P(X > 2)$. That is, given we've waited 7 minutes, the probability we wait at least 2 more, is the same as the probability we wait at least 2 more from the beginning.

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The only memoryless RVs are the **Geometric** (discrete) and **Exponential** (continuous)!

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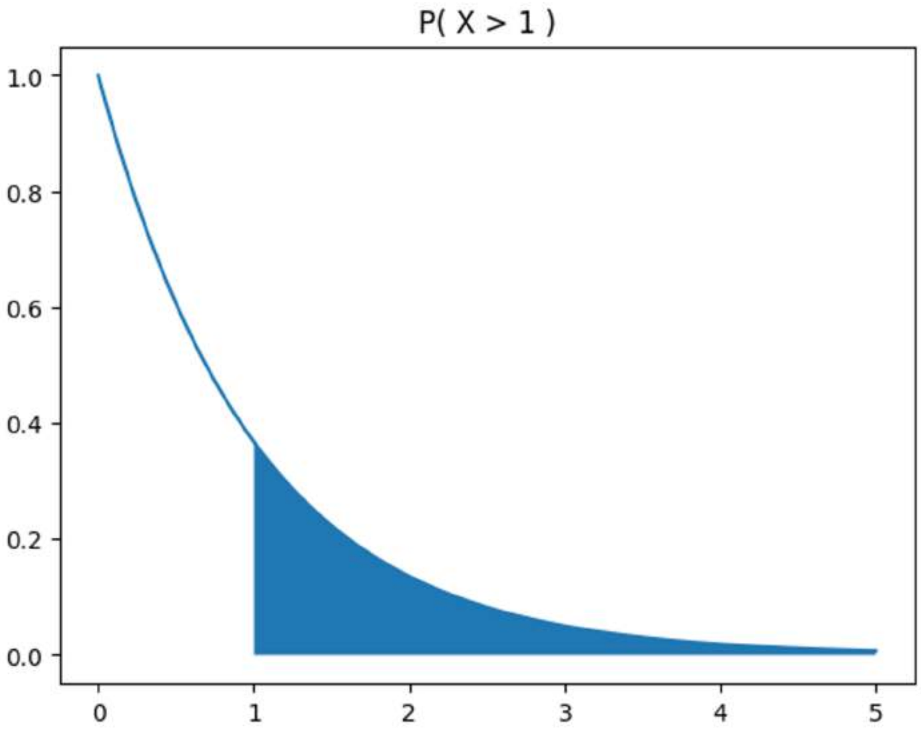

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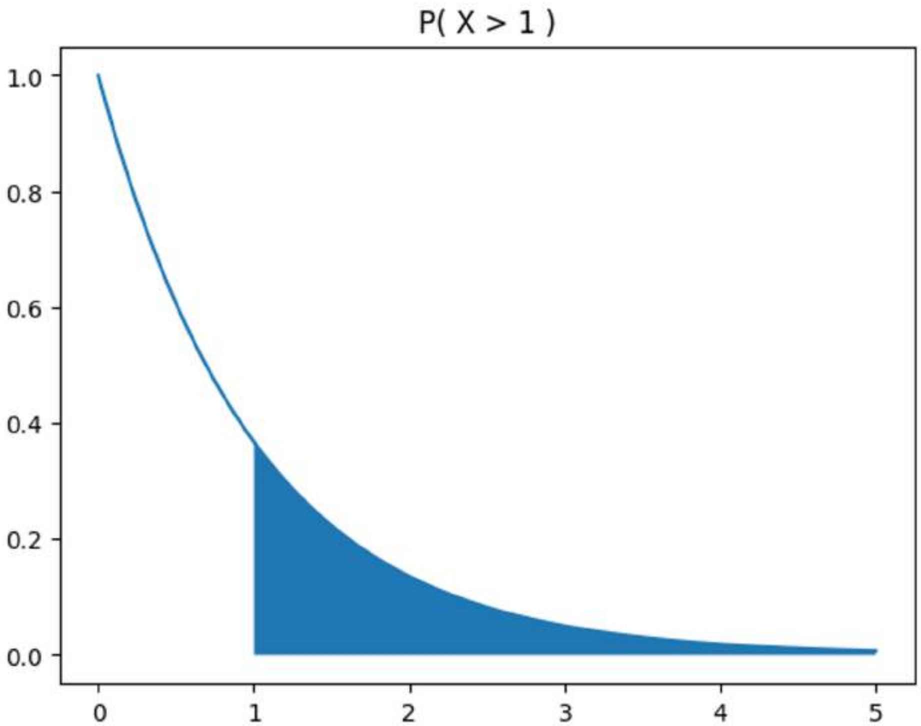


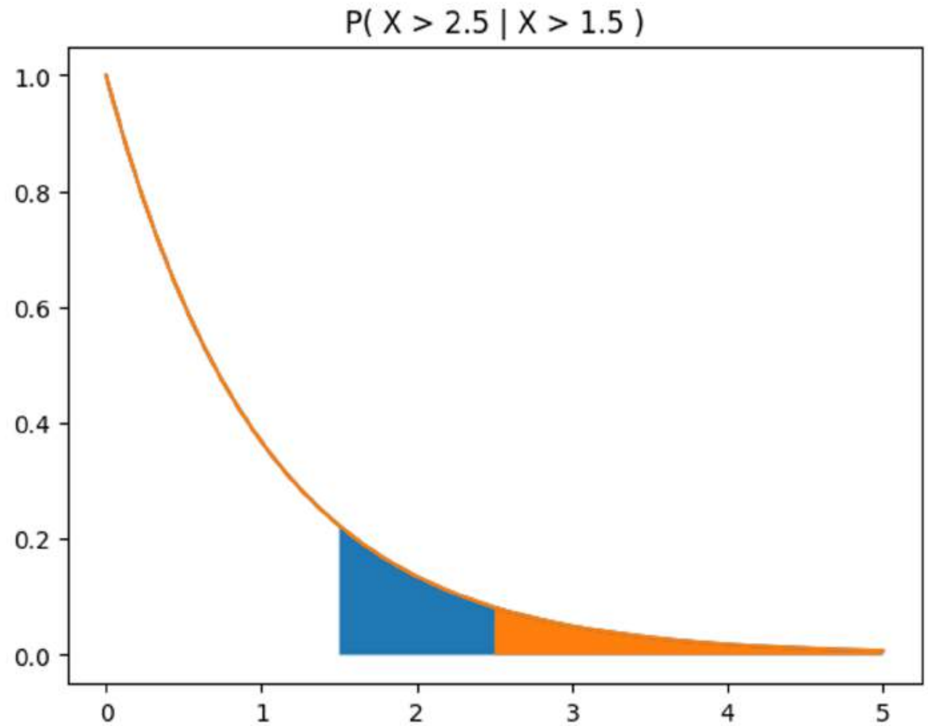

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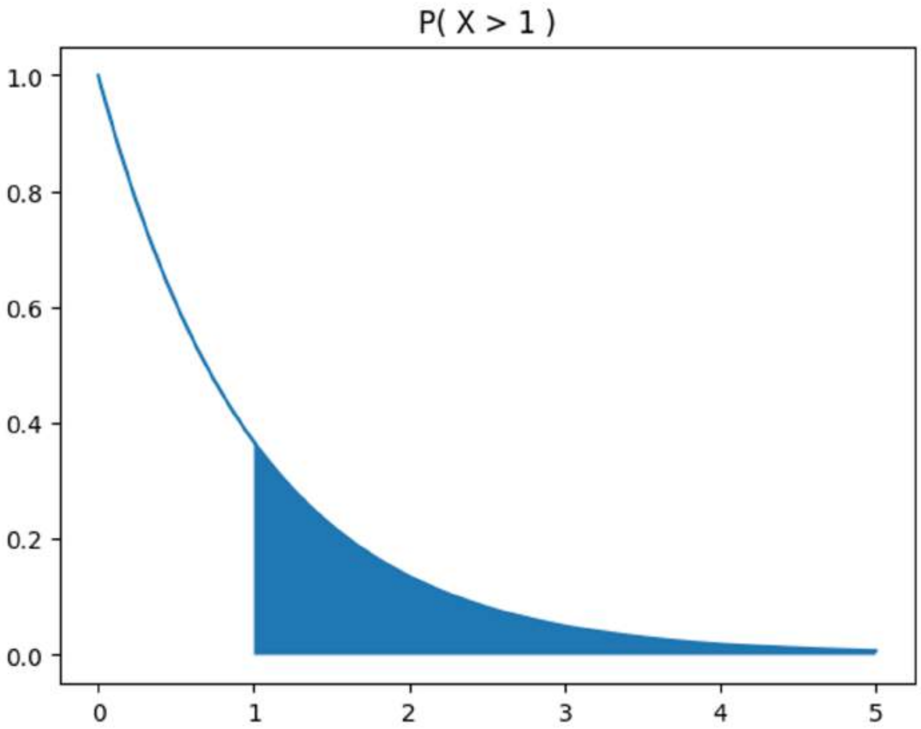


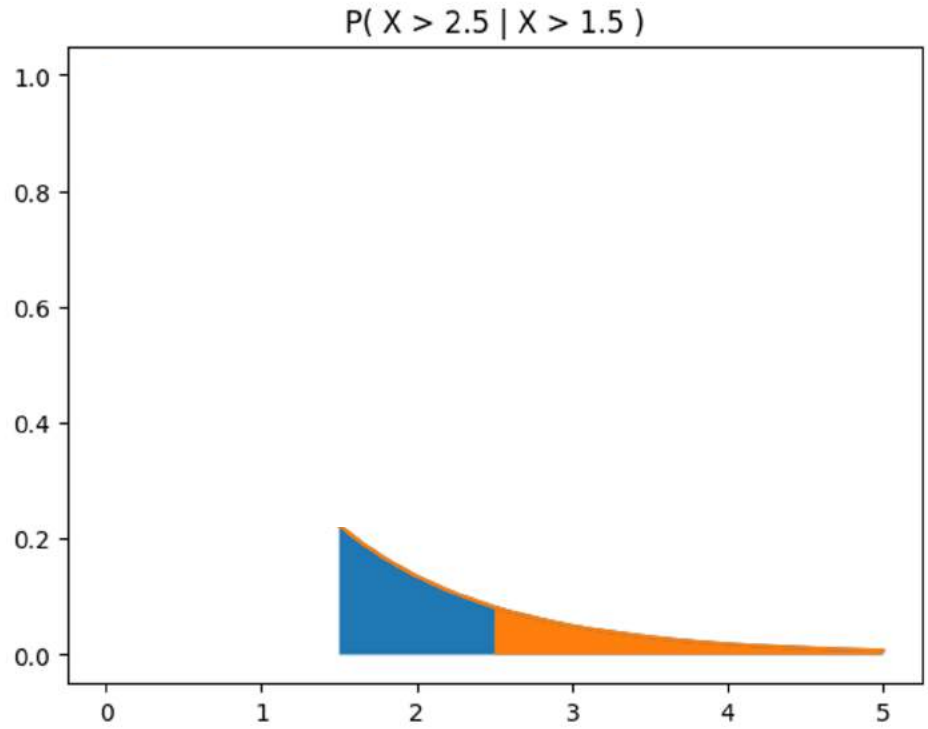

# Memorylessness of Exponential (Proof)

If $X \sim Exp(\lambda)$ and $x \geq 0$, then recall

$$P(X > x) = 1 - F_X(x) = 1 - (1 - e^{-\lambda x}) = e^{-\lambda x}$$

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$$\begin{aligned} P(X > s + t \mid X > s) &= \frac{P(X > s \mid X > s + t) P(X > s + t)}{P(X > s)} \\ &= \frac{P(X > s + t)}{P(X > s)} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \\ &= e^{-\lambda t} \\ &= P(X > t) \end{aligned}$$

# EXAMPLE

- Time it takes to check someone out at a grocery store is exponential with an expected value of 10 mins.
- Independent for different customers.
- If you are the second person in line, what is the probability that you will have to wait between 10 and 20 mins.

# EXAMPLE

- Time it takes to check someone out at a grocery store is exponential with an expected value of 10 mins.
- Independent for different customers
- If you are the second person in line, what is the probability that you will have to wait between 10 and 20 mins.

$$T \sim exp(10^{-1})$$

$$Pr(10 \leq T \leq 20) = \int_{10}^{20} \frac{1}{10} e^{-\frac{x}{10}} dx$$

$$y = \frac{x}{10} \qquad dy = \frac{1}{10} dx$$

$$Pr(10 \leq T \leq 20) = \frac{1}{10} \int_{1}^{2} e^{-y} dy = -\frac{1}{10} e^{-y} \Big|_1^2 = \frac{1}{10}(e^{-1} - e^{-2})$$

# 4.3 The Normal/Gaussian Random Variable

# Agenda

- The Normal/Gaussian RV
- Closure properties of the Normal RV
- The standard normal CDF

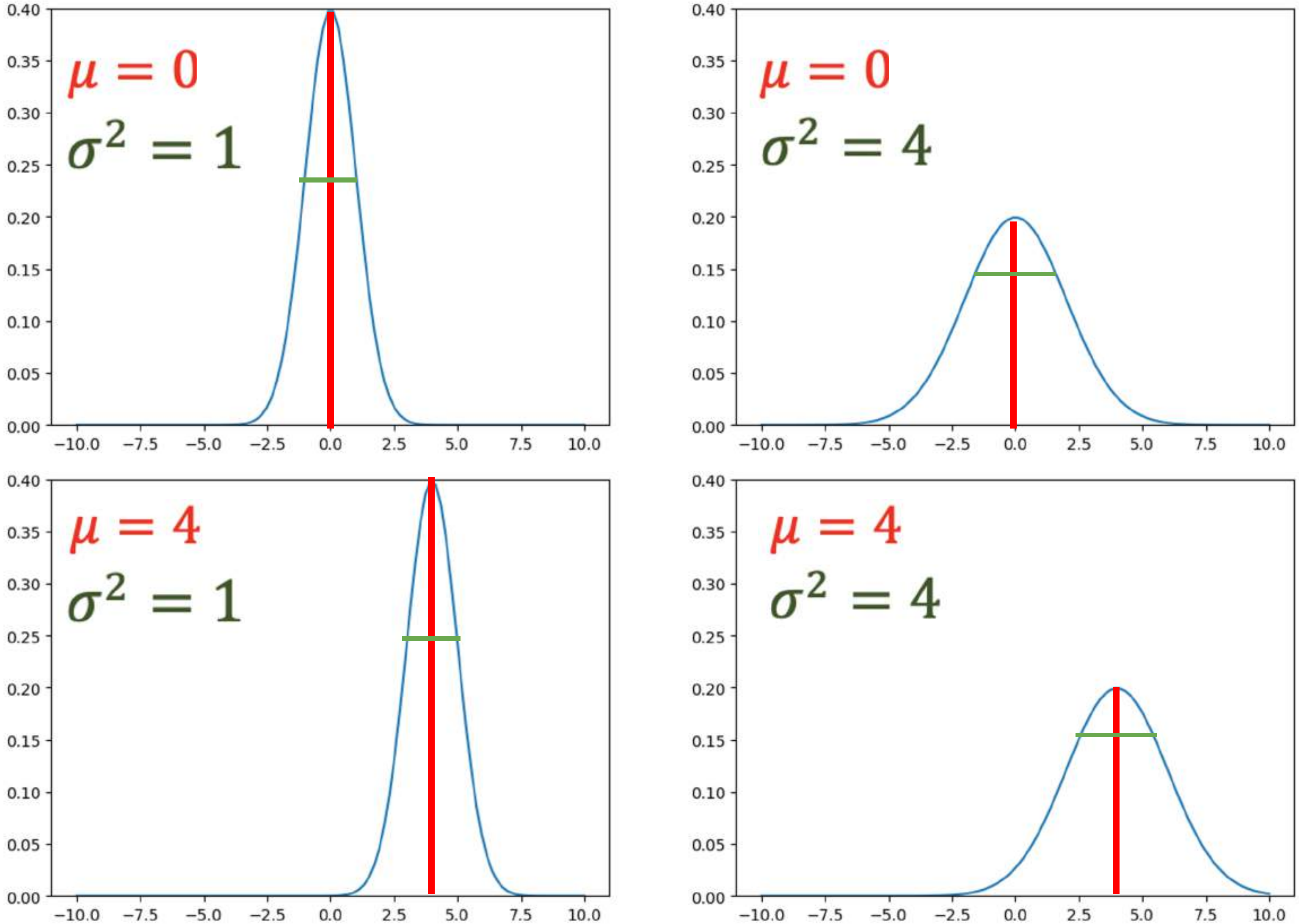
# The Normal/Gaussian RV

**Normal (Gaussian, "bell curve") Distribution:** $X \sim \mathcal{N}(\mu, \sigma^2)$ if and only if $X$ has the following pdf:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$E[X] = \mu \qquad\qquad Var(X) = \sigma^2$$

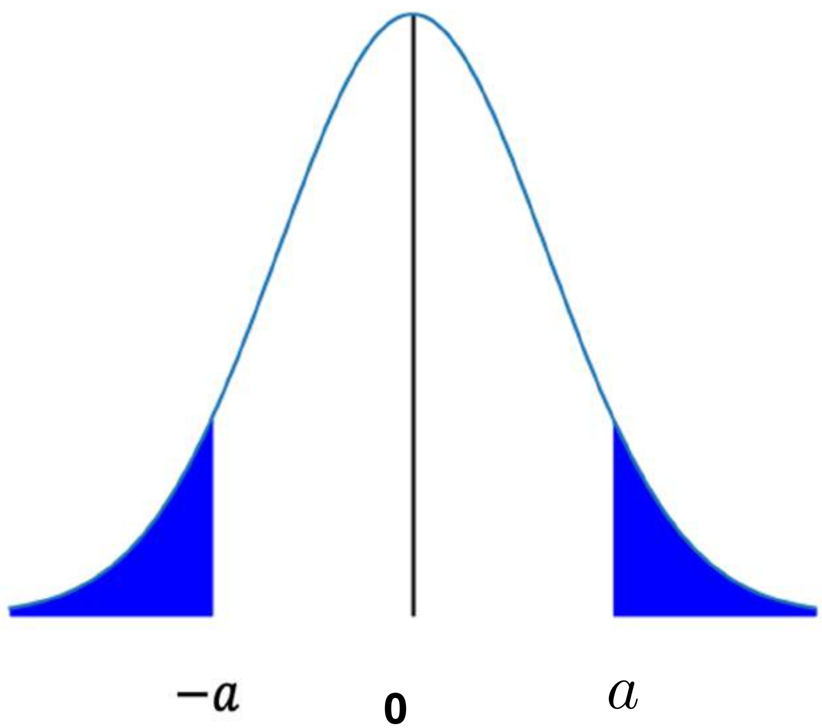
# THE NORMAL PDF



# Random Picture

# The Standard Normal CDF

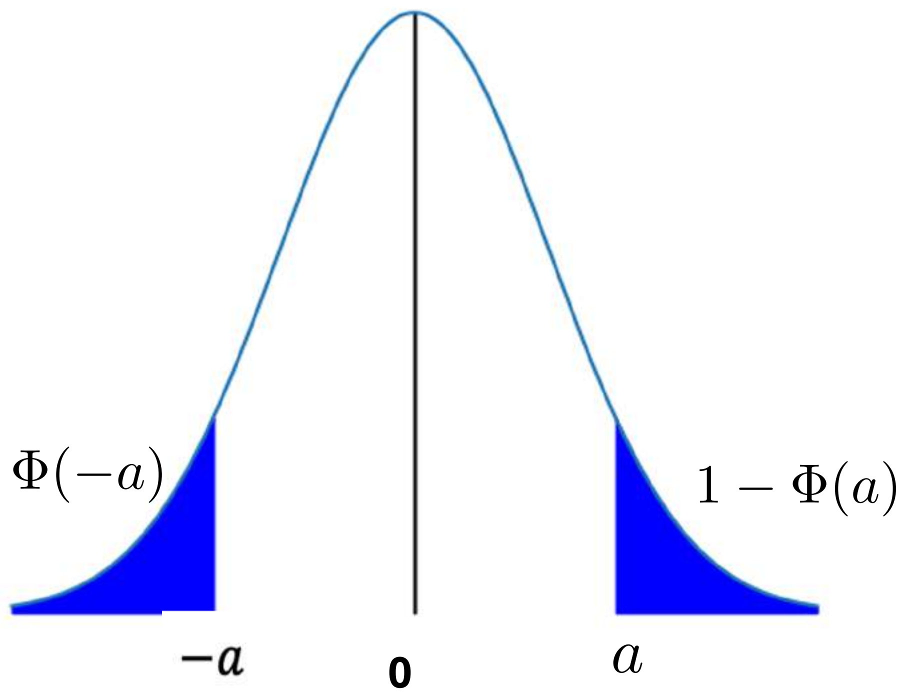
$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



# The Standard Normal CDF

If $Z \sim \mathcal{N}(0,1)$, we denote the CDF $\Phi(a) = F_Z(a) = P(Z \leq a)$, since it's so commonly used. There is no closed-form formula, so this CDF is stored in a $\Phi$ table.

$$\Phi(-a) = 1 - \Phi(a)$$



# The Standard Normal CDF

$$P(Z \leq 1.09) = \Phi(1.09) \approx 0.8621$$

$\Phi$ Table: $\mathbb{P}(Z \leq z)$ when $Z \sim \mathcal{N}(0,1)$

| $z$ | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.5 | 0.50399 | 0.50798 | 0.51197 | 0.51595 | 0.51994 | 0.52392 | 0.5279 | 0.53188 | 0.53586 |
| 0.1 | 0.53983 | 0.5438 | 0.54776 | 0.55172 | 0.55567 | 0.55962 | 0.56356 | 0.56749 | 0.57142 | 0.57535 |
| 0.2 | 0.57926 | 0.58317 | 0.58706 | 0.59095 | 0.59483 | 0.59871 | 0.60257 | 0.60642 | 0.61026 | 0.61409 |
| 0.3 | 0.61791 | 0.62172 | 0.62552 | 0.6293 | 0.63307 | 0.63683 | 0.64058 | 0.64431 | 0.64803 | 0.65173 |
| 0.4 | 0.65542 | 0.6591 | 0.66276 | 0.6664 | 0.67003 | 0.67364 | 0.67724 | 0.68082 | 0.68439 | 0.68793 |
| 0.5 | 0.69146 | 0.69497 | 0.69847 | 0.70194 | 0.7054 | 0.70884 | 0.71226 | 0.71566 | 0.71904 | 0.7224 |
| 0.6 | 0.72575 | 0.72907 | 0.73237 | 0.73565 | 0.73891 | 0.74215 | 0.74537 | 0.74857 | 0.75175 | 0.7549 |
| 0.7 | 0.75804 | 0.76115 | 0.76424 | 0.7673 | 0.77035 | 0.77337 | 0.77637 | 0.77935 | 0.7823 | 0.78524 |
| 0.8 | 0.78814 | 0.79103 | 0.79389 | 0.79673 | 0.79955 | 0.80234 | 0.80511 | 0.80785 | 0.81057 | 0.81327 |
| 0.9 | 0.81594 | 0.81859 | 0.82121 | 0.82381 | 0.82639 | 0.82894 | 0.83147 | 0.83398 | 0.83646 | 0.83891 |
| 1.0 | 0.84134 | 0.84375 | 0.84614 | 0.84849 | 0.85083 | 0.85314 | 0.85543 | 0.85769 | 0.85993 | 0.86214 |
| 1.1 | 0.86433 | 0.8665 | 0.86864 | 0.87076 | 0.87286 | 0.87493 | 0.87698 | 0.879 | 0.881 | 0.88298 |
| 1.2 | 0.88493 | 0.88686 | 0.88877 | 0.89065 | 0.89251 | 0.89435 | 0.89617 | 0.89796 | 0.89973 | 0.90147 |
| 1.3 | 0.9032 | 0.9049 | 0.90658 | 0.90824 | 0.90988 | 0.91149 | 0.91309 | 0.91466 | 0.91621 | 0.91774 |
| 1.4 | 0.91924 | 0.92073 | 0.9222 | 0.92364 | 0.92507 | 0.92647 | 0.92785 | 0.92922 | 0.93056 | 0.93189 |
| 1.5 | 0.93319 | 0.93448 | 0.93574 | 0.93699 | 0.93822 | 0.93943 | 0.94062 | 0.94179 | 0.94295 | 0.94408 |
| 1.6 | 0.9452 | 0.9463 | 0.94738 | 0.94845 | 0.9495 | 0.95053 | 0.95154 | 0.95254 | 0.95352 | 0.95449 |
| 1.7 | 0.95543 | 0.95637 | 0.95728 | 0.95818 | 0.95907 | 0.95994 | 0.9608 | 0.96164 | 0.96246 | 0.96327 |
| 1.8 | 0.96407 | 0.96485 | 0.96562 | 0.96638 | 0.96712 | 0.96784 | 0.96856 | 0.96926 | 0.96995 | 0.97062 |
| 1.9 | 0.97128 | 0.97193 | 0.97257 | 0.9732 | 0.97381 | 0.97441 | 0.975 | 0.97558 | 0.97615 | 0.9767 |
| 2.0 | 0.97725 | 0.97778 | 0.97831 | 0.97882 | 0.97932 | 0.97982 | 0.9803 | 0.98077 | 0.98124 | 0.98169 |
| 2.1 | 0.98214 | 0.98257 | 0.983 | 0.98341 | 0.98382 | 0.98422 | 0.98461 | 0.985 | 0.98537 | 0.98574 |
| 2.2 | 0.9861 | 0.98645 | 0.98679 | 0.98713 | 0.98745 | 0.98778 | 0.98809 | 0.9884 | 0.9887 | 0.98899 |
| 2.3 | 0.98928 | 0.98956 | 0.98983 | 0.9901 | 0.99036 | 0.99061 | 0.99086 | 0.99111 | 0.99134 | 0.99158 |
| 2.4 | 0.9918 | 0.99202 | 0.99224 | 0.99245 | 0.99266 | 0.99286 | 0.99305 | 0.99324 | 0.99343 | 0.99361 |
| 2.5 | 0.99379 | 0.99396 | 0.99413 | 0.9943 | 0.99446 | 0.99461 | 0.99477 | 0.99492 | 0.99506 | 0.9952 |
| 2.6 | 0.99534 | 0.99547 | 0.9956 | 0.99573 | 0.99585 | 0.99598 | 0.99609 | 0.99621 | 0.99632 | 0.99643 |
| 2.7 | 0.99653 | 0.99664 | 0.99674 | 0.99683 | 0.99693 | 0.99702 | 0.99711 | 0.9972 | 0.99728 | 0.99736 |
| 2.8 | 0.99744 | 0.99752 | 0.9976 | 0.99767 | 0.99774 | 0.99781 | 0.99788 | 0.99795 | 0.99801 | 0.99807 |
| 2.9 | 0.99813 | 0.99819 | 0.99825 | 0.99831 | 0.99836 | 0.99841 | 0.99846 | 0.99851 | 0.99856 | 0.99861 |
| 3.0 | 0.99865 | 0.99869 | 0.99874 | 0.99878 | 0.99882 | 0.99886 | 0.99889 | 0.99893 | 0.99896 | 0.999 |

# What about non-Standard Normals?

$X \sim \mathcal{N}(\mu, \sigma^2)$,

# We can Standardize any RV

Let $X$ be **ANY** random variable (discrete or continuous) with $E[X] = \mu$ and $Var(X) = \sigma^2$ , and $a, b \in \mathbb{R}$. Then,

$$E[aX + b] = aE[X] + b = a\mu + b$$

$$Var(aX + b) = a^2 Var(X) = a^2\sigma^2$$

In particular, we call $\frac{X-\mu}{\sigma}$ a standardized version of $X$, as it measures how many standard deviations above the mean a point is.

$$E\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma}(E[X] - \mu) = 0$$

$$Var\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2} Var(X - \mu) = \frac{1}{\sigma^2}\sigma^2 = 1$$

# Normals stay normal! (Under scale+Shift)

# Closure of the normal (Under scale+Shift)


Let $X$ be **ANY** random variable (discrete or continuous) with $E[X] = \mu$ and $Var(X) = \sigma^2$ , and $a, b \in \mathbb{R}$. Recall,

$$E[aX + b] = aE[X] + b = a\mu + b$$

$$Var(aX + b) = a^2Var(X) = a^2\sigma^2$$

But if $X \sim \mathcal{N}(\mu, \sigma^2)$ (a Normal rv), then

$$aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$

In particular,

$$\frac{X - \mu}{\sigma} \sim \mathcal{N}(0,1)$$

Note the "special" thing here is that the transformed RV remains a Normal rv – the mean and variance are no surprise.

X is normal with mean 3 and variance 9. What is

- $\Pr(2 < X < 5)$?
- $\Pr(X > 0)$?
- $\Pr(|X-3| > 6)$?

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What is

- Pr (2 < X < 5)?
- Pr (X > 0)?
- Pr (|X-3| > 6)?

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| 1.3 | 0.9032 | 0.9049 | 0.90658 | 0.90824 | 0.90988 | 0.91149 | 0.91309 | 0.91466 | 0.91621 | 0.91774 |
| 1.4 | 0.91924 | 0.92073 | 0.9222 | 0.92364 | 0.92507 | 0.92647 | 0.92785 | 0.92922 | 0.93056 | 0.93189 |
| 1.5 | 0.93319 | 0.93448 | 0.93574 | 0.93699 | 0.93822 | 0.93943 | 0.94062 | 0.94179 | 0.94295 | 0.94408 |
| 1.6 | 0.9452 | 0.9463 | 0.94738 | 0.94845 | 0.9495 | 0.95053 | 0.95154 | 0.95254 | 0.95352 | 0.95449 |
| 1.7 | 0.95543 | 0.95637 | 0.95728 | 0.95818 | 0.95907 | 0.95994 | 0.9608 | 0.96164 | 0.96246 | 0.96327 |
| 1.8 | 0.96407 | 0.96485 | 0.96562 | 0.96638 | 0.96712 | 0.96784 | 0.96856 | 0.96926 | 0.96995 | 0.97062 |
| 1.9 | 0.97128 | 0.97193 | 0.97257 | 0.9732 | 0.97381 | 0.97441 | 0.975 | 0.97558 | 0.97615 | 0.9767 |
| 2.0 | 0.97725 | 0.97778 | 0.97831 | 0.97882 | 0.97932 | 0.97982 | 0.9803 | 0.98077 | 0.98124 | 0.98169 |
| 2.1 | 0.98214 | 0.98257 | 0.983 | 0.98341 | 0.98382 | 0.98422 | 0.98461 | 0.985 | 0.98537 | 0.98574 |
| 2.2 | 0.9861 | 0.98645 | 0.98679 | 0.98713 | 0.98745 | 0.98778 | 0.98809 | 0.9884 | 0.9887 | 0.98899 |
| 2.3 | 0.98928 | 0.98956 | 0.98983 | 0.9901 | 0.99036 | 0.99061 | 0.99086 | 0.99111 | 0.99134 | 0.99158 |
| 2.4 | 0.9918 | 0.99202 | 0.99224 | 0.99245 | 0.99266 | 0.99286 | 0.99305 | 0.99324 | 0.99343 | 0.99361 |
| 2.5 | 0.99379 | 0.99396 | 0.99413 | 0.9943 | 0.99446 | 0.99461 | 0.99477 | 0.99492 | 0.99506 | 0.9952 |
| 2.6 | 0.99534 | 0.99547 | 0.9956 | 0.99573 | 0.99585 | 0.99598 | 0.99609 | 0.99621 | 0.99632 | 0.99643 |
| 2.7 | 0.99653 | 0.99664 | 0.99674 | 0.99683 | 0.99693 | 0.99702 | 0.99711 | 0.9972 | 0.99728 | 0.99736 |
| 2.8 | 0.99744 | 0.99752 | 0.9976 | 0.99767 | 0.99774 | 0.99781 | 0.99788 | 0.99795 | 0.99801 | 0.99807 |
| 2.9 | 0.99813 | 0.99819 | 0.99825 | 0.99831 | 0.99836 | 0.99841 | 0.99846 | 0.99851 | 0.99856 | 0.99861 |
| 3.0 | 0.99865 | 0.99869 | 0.99874 | 0.99878 | 0.99882 | 0.99886 | 0.99889 | 0.99893 | 0.99896 | 0.999 |

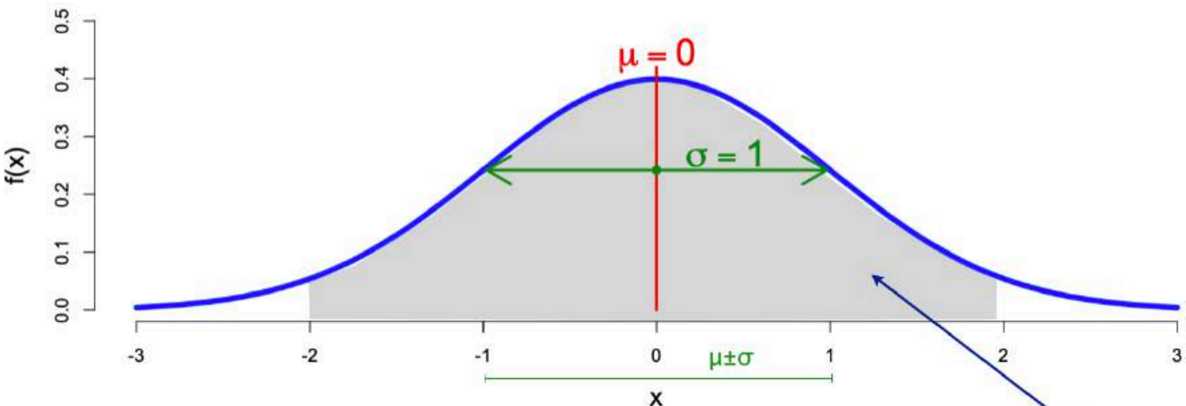
# FROM $N(\mu, \sigma^2)$ TO STANDARD NORMAL


For a $X \sim \mathcal{N}(\mu, \sigma^2)$, we have

$$F_X(y) = P(X \leq y) = P\left(\frac{X-\mu}{\sigma} \leq \frac{y-\mu}{\sigma}\right) = P\left(Z \leq \frac{y-\mu}{\sigma}\right) = \Phi\left(\frac{y-\mu}{\sigma}\right)$$

$$P(a \leq X \leq b) = F_X(b) - F_X(a) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

# Normal random variables




If $Z \sim N(\mu,\sigma^2)$ what is $P(\mu-\sigma < Z < \mu+\sigma)$?

$P(\mu - \sigma < Z < \mu + \sigma) = \Phi(1) - \Phi(-1) \approx 68\%$

$P(\mu - 2\sigma < Z < \mu + 2\sigma) = \Phi(2) - \Phi(-2) \approx 95\%$

$P(\mu - 3\sigma < Z < \mu + 3\sigma) = \Phi(3) - \Phi(-3) \approx 99\%$

Why?

$N(\mu,\sigma^2)$     $N(0,1)$

$\mu - k\sigma < \boxed{Z} < \mu + k\sigma \Leftrightarrow -k < \boxed{(Z-\mu)/\sigma} < +k$

# Summary: The Normal/Gaussian RV

**Normal (Gaussian, "bell curve") Distribution:** $X \sim \mathcal{N}(\mu, \sigma^2)$ if and only if $X$ has the following pdf:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$E[X] = \mu \qquad\qquad Var(X) = \sigma^2$$

The "standard normal" random variable is typically denoted $Z$ and has mean $0$ and variance $1$. By the closure property of normals, if $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$. The CDF has no closed form, but we denote the CDF of the standard normal by $\Phi(a) = F_Z(a) = P(Z \leq a)$. Note that by symmetry of the density about $0$, $\Phi(-a) = 1 - \Phi(a)$.

# Closure of the normal (under addition)


Let $X, Y$ be **ANY independent** random variables (discrete or continuous) with $E[X] = \mu_X$, $E[Y] = \mu_Y$, $Var(X) = \sigma_X^2$ , $Var(Y) = \sigma_Y^2$, and $a, b, c \in \mathbb{R}$. Recall,

$$E[aX + bY + c] = aE[X] + bE[Y] + c = a\mu_X + b\mu_Y + c$$

$$Var(aX + bY + c) = a^2 Var(X) + b^2 Var(Y) = a^2\sigma_X^2 + b^2\sigma_Y^2$$

But if $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ (both independent Normal rvs), then

$$aX + bY + c \sim \mathcal{N}(a\mu_X + b\mu_Y + c, a^2\sigma_X^2 + b^2\sigma_Y^2)$$

Note the "special" thing here is that the sum remains a Normal rv – the mean and variance are no surprise.

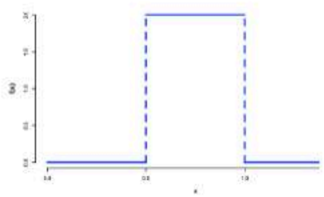
# Summary: zoo of continuous random variables

Three important examples

## X ~ Uni(α,β) uniform in [α,β]

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & x \in [\alpha, \beta] \\ 0 & \text{otherwise} \end{cases}$$

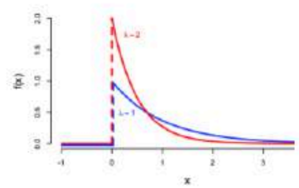
E[X] = (α+β)/2
Var[X] = $(\alpha-\beta)^2/12$



## X ~ Exp(λ) exponential

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

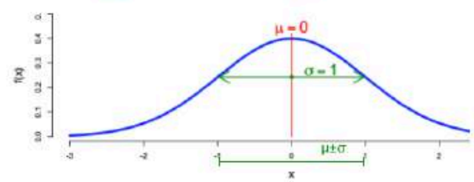
$$E[X] = \frac{1}{\lambda}$$
$$\mathrm{Var}[X] = \frac{1}{\lambda^2}$$

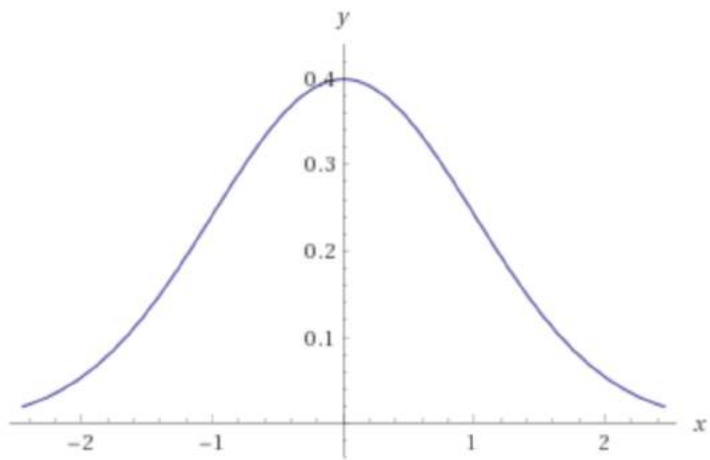


## X ~ N(μ, σ²) normal (aka Gaussian, aka the big Kahuna)

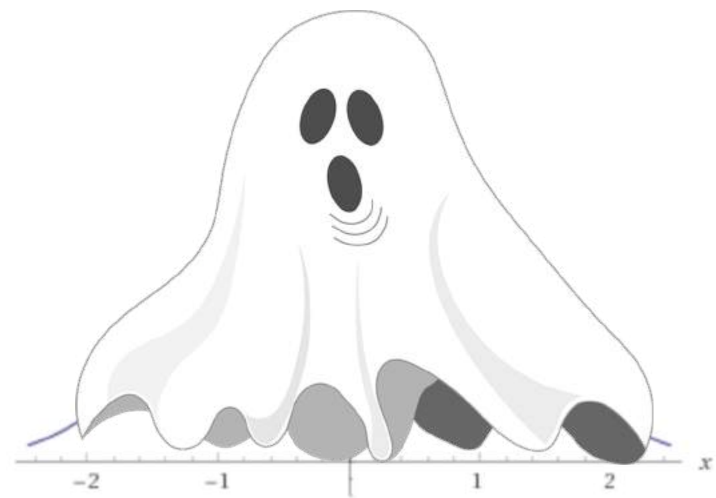
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

$$E[X] = \mu$$
$$\mathrm{Var}[X] = \sigma^2$$






Normal Distribution




Paranormal Distribution