Continuous random Variables

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Most Slides by Alex Tsun + Joshua Fan

Agenda

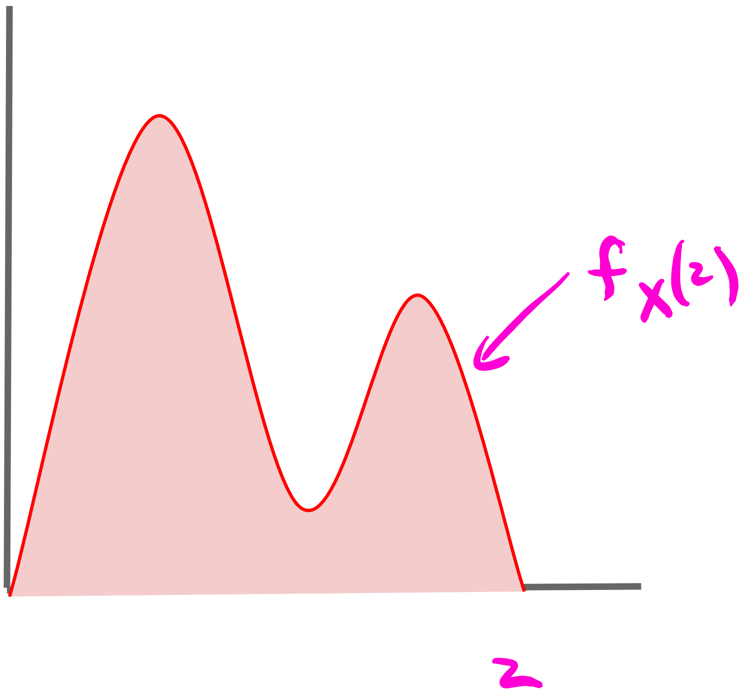
- Recap (PDFs and CDFs)
- The (Continuous) Uniform RV
- The Exponential RV
- Memorylessness
- The normal distribution

From Discrete to Continuous

	Discrete	Continuous
PMF/PDF	$p_X(x) = \mathbb{P}(X = x)$	$f_X(x) \neq \mathbb{P}(X = x) = 0$
CDF	$F_X(x) = \sum_{t \leq x} p_X(t)$	$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$
Normalization	$\sum_x p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$
Expectation	$\mathbb{E}[X] = \sum_x x p_X(x)$	$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$
LOTUS	$\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$

X cont r.v.

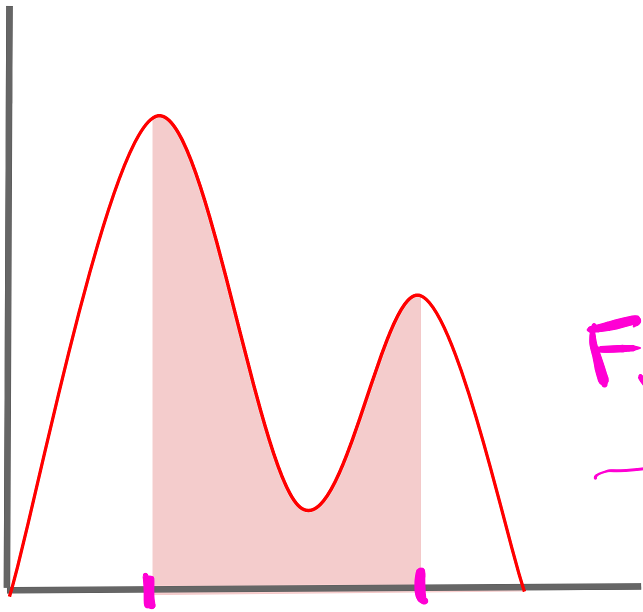
PDF Intuition



$$f_X(z) \geq 0 \text{ for all } z \in \mathbb{R}$$

$$\int_{-\infty}^{\infty} f_X(t)\,dt = 1$$

PDF Intuition



$$f_X(z) \geq 0 \text{ for all } z \in \mathbb{R}$$

$$\int_{-\infty}^{\infty} f_X(t)dt = 1$$

$$P(a \leq X \leq b) = \int_a^b f_X(w)dw$$

$$F_X(x) = Pr(X \leq x) = \int_{-\infty}^{x} f_X(w)\,dw$$

$$f_X(x) = \frac{d}{dx} F_X(x)$$

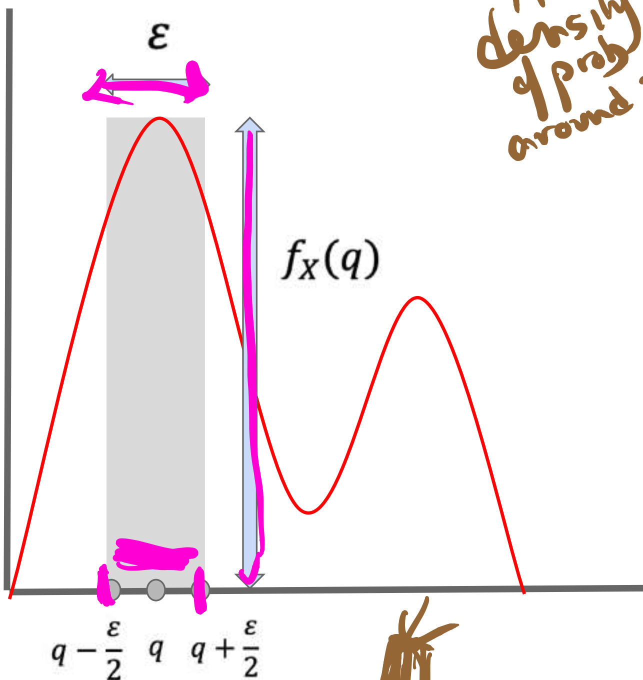
$$Pr(X \leq x) - Pr(X < x) = Pr(X = x) = 0$$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

$$f_X(z) \neq \Pr[X=z]=0$$

density of prob around z.

PDF Intuition



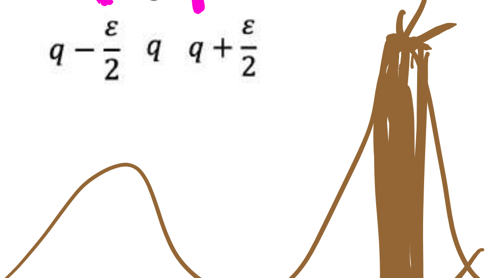
$f_X(z) \geq 0$ for all $z \in \mathbb{R}$

$$\boxed{\int_{-\infty}^{\infty} f_X(t)dt = 1}$$

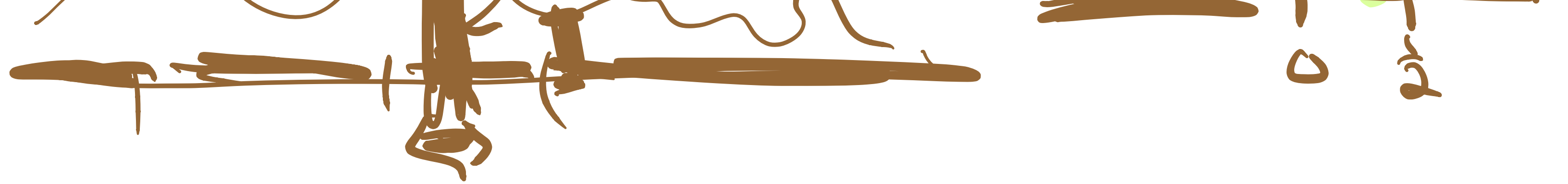
$$P(a \leq X \leq b) = \int_a^b f_X(w)dw$$

$$P(X = y) = P(y \leq X \leq y) = \boxed{\int_y^y f_X(w)dw} = 0$$

$$P(X \approx q) \approx P\left(q - \frac{\varepsilon}{2} \leq X \leq q + \frac{\varepsilon}{2}\right) \approx \varepsilon f_X(q)$$



Unif$(0, \frac{1}{2})$



Cumulative Distribution Functions (CDFs)

Cumulative Distribution Function (CDF): Let X be a continuous rv (one whose range is typically an interval or union of intervals). The cumulative distribution function (CDF) of X is the function $F_X: \mathbb{R} \to \mathbb{R}$ such that

- $F_X(t) = P(X \leq t) = \int_{-\infty}^{t} f_X(w)dw$ for all $t \in \mathbb{R}$.
- Hence, by the Fundamental Theorem of Calculus, $\frac{d}{du}F_X(u) = f_X(u)$.
- $P(a \leq X \leq b) = F_X(b) - F_X(a)$.
- F_X is monotone increasing, since $f_X \geq 0$. That is, $F_X(c) \leq F_X(d)$ for $c \leq d$.
- $\lim\limits_{v \to -\infty} F_X(v) = P(X \leq -\infty) = 0$.
- $\lim\limits_{v \to +\infty} F_X(v) = P(X \leq +\infty) = 1$.

4.2 Zoo of Continuous RVs

Unif$\{a, a+1, \ldots, b\}$

The Uniform (Continuous) RV

Uniform (Continuous) RV: $X \sim Unif(a, b)$ where $a < b$ are real numbers, if and only if X has the following pdf:

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & x \in [a, b] \\ 0, & \text{otherwise} \end{cases}$$

Unif$\{0, 1\}$

$\frac{1}{2}$ $\frac{1}{2}$

Unif$(0, 1)$

X is equally likely to take on any value in $[a, b]$.

$$E[X] = \frac{a+b}{2} \qquad \boxed{Var(X) = \frac{(b-a)^2}{12}}$$

The cdf is

$$F_X(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \le x \le b \\ 1, & x > b \end{cases}$$

The Exponential PDF/CDF

Recall the Poisson Process with parameter $\lambda > 0$ has events happening at average rate of λ per unit of time forever. The exponential RV measures the time until the first occurrence of an event, so is a continuous RV with range $[0, \infty)$ (unlike the Poisson RV, which counts the number of occurrences in a unit of time, with range $\{0,1,2,\ldots\}$.)

t

events occur in t timeunits

$\boxed{X(t)} \sim Poi(\lambda t)$

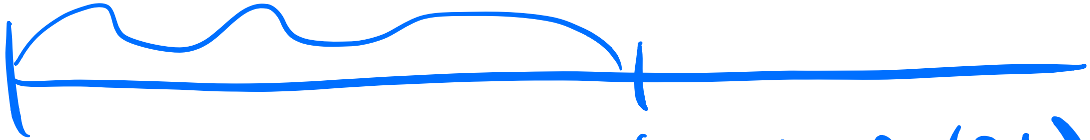
events occur rate λ / timeunits

events in disjoint intervals indep

Exp: time I wait until first event.

X

1 hour



t $X(t) \sim Poi(\lambda t)$

$Pr(X(t)=k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$

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Let $\boxed{Y \sim Exp(\lambda)}$ be the time until the first event. We'll compute $F_Y(t)$ and $f_Y(t)$.
Let $X(t) \sim Poi(\lambda t)$ be the # of events in the first t units of time, for $t \geq 0$.

$$Pr(Y > t) = Pr(X(t) = 0)$$

$$= e^{-\lambda t}$$

$$F_Y(t) = Pr(Y \leq t) = 1 - Pr(Y > t) = 1 - e^{-\lambda t} \quad t \geq 0$$

$$F_Y(t) = 0 \quad t < 0$$

The Exponential PDF/CDF

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Let $Y \sim Exp(\lambda)$ be the time until the first event. We'll compute $F_Y(t)$ and $f_Y(t)$.
Let $X(t) \sim Poi(\lambda t)$ be the # of events in the first t units of time, for $t \geq 0$.

$$P(Y > t) = P(\text{no events in first } t \text{ units}) = P(X(t) = 0) = e^{-\lambda t}\frac{(\lambda t)^0}{0!} = e^{-\lambda t}$$

$$\boxed{F_Y(t) = P(Y \leq t) = 1 - P(Y > t) = 1 - e^{-\lambda t}}$$

$t > 0$

$F_Y(t) = 0$ $t < 0$

$$f_Y(t) = \frac{d}{dt}F_Y(t) = \lambda e^{-\lambda t}$$

$t \geq 0$

0 $t < 0$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

The Exponential RV Properties

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \int_0^\infty x \lambda e^{-\lambda x} dx = \boxed{\frac{1}{\lambda}}$$

$$Var(X) = E(X^2) - [E(X)]^2.$$

5 per min.

$$\Downarrow$$

$$\int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_0^\infty x^2 \lambda e^{-\lambda x} dx = \frac{2}{\lambda^2}$$

The Exponential RV Properties

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \int_{0}^{\infty} x \cdot \lambda e^{-\lambda x} dx = \frac{1}{\lambda}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx = \int_{0}^{\infty} x^2 \cdot \lambda e^{-\lambda x} dx = \boxed{\frac{2}{\lambda^2}}$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

The Exponential RV

Exponential RV: $X \sim Exp(\lambda)$, if and only if X has the following pdf:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

X is the waiting time until the first occurrence of an event in a Poisson process with parameter λ.

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X is the waiting time until the first occurrence of an event in a Poisson process with parameter λ.

$$E[X] = \frac{1}{\lambda} \qquad Var(X) = \frac{1}{\lambda^2}$$

The cdf is

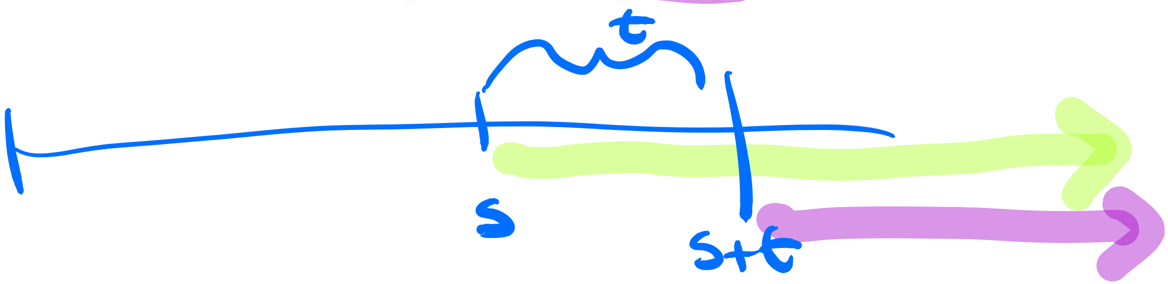
$$F_X(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Random Picture

Memorylessness (Intuition)

A random variable X is **memoryless** if for all $s, t \geq 0$,

$$P(X > s + t \mid X > s) = P(X > t)$$



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A random variable X is **memoryless** if for all $s, t \geq 0$,

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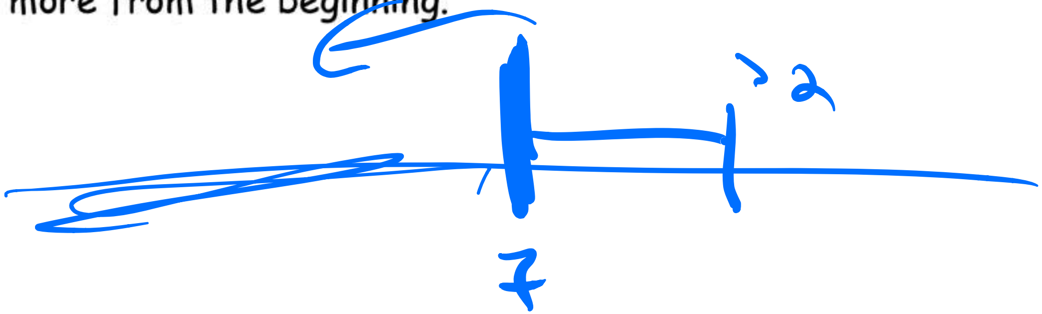
For example, let $s = 7, t = 2$. So $P(X > 9 \mid X > 7) = P(X > 2)$. That is, given we've waited 7 minutes, the probability we wait at least 2 more, is the same as the probability we wait at least 2 more from the beginning.



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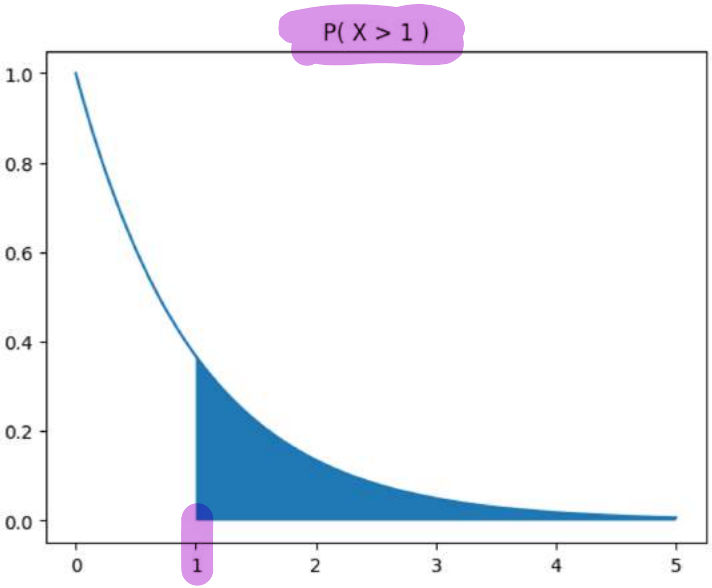
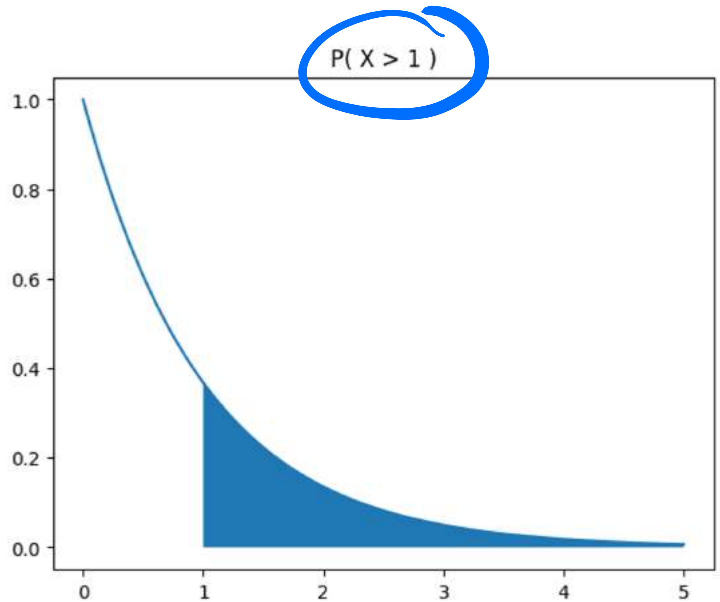
The only memoryless RVs are the **Geometric** (discrete) and **Exponential** (continuous)!

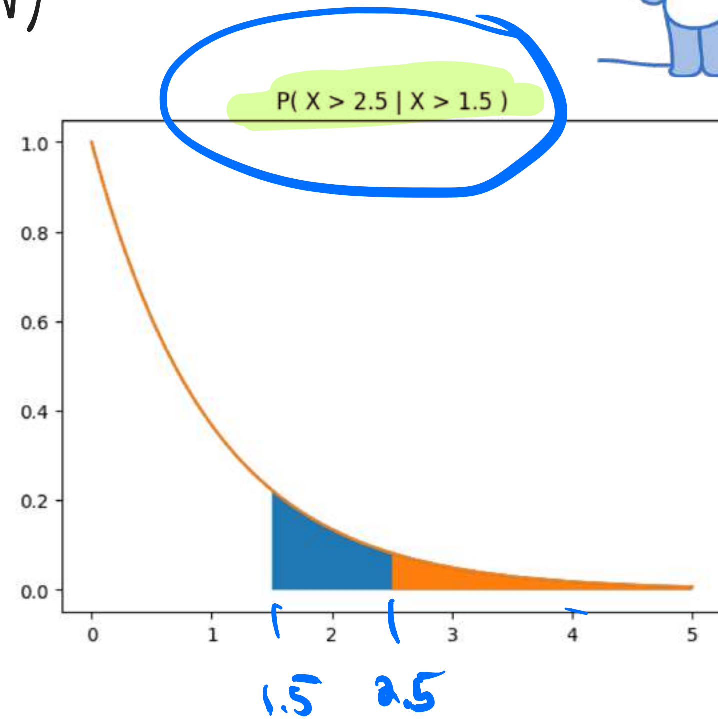
Memorylessness (Intuition)



$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Memorylessness (Intuition)

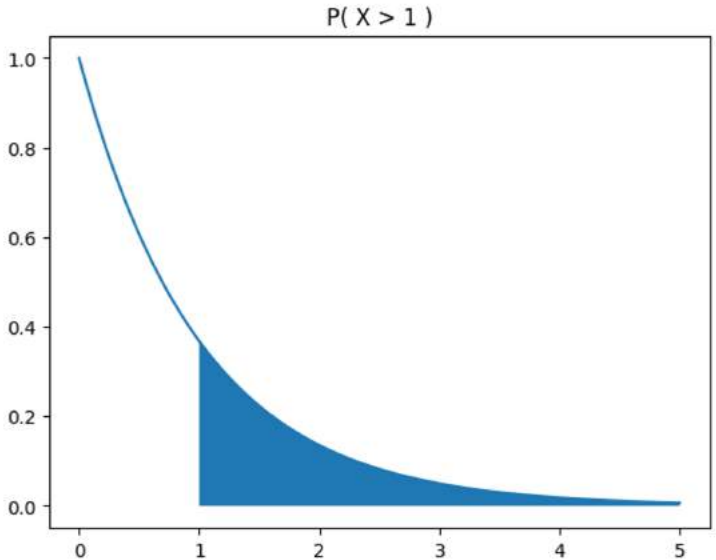


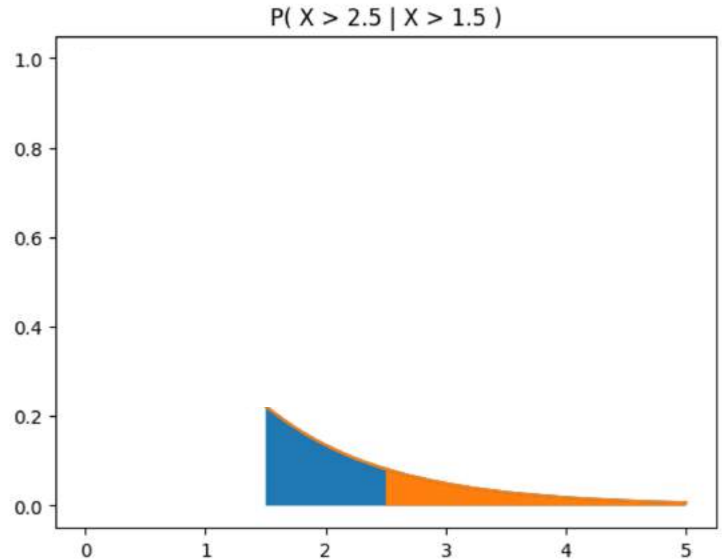


$$\frac{\text{blue}}{\text{total area}} = \frac{\text{orange}}{\text{orange} + \text{blue}}$$

total area $=$ 1

Memorylessness (Intuition)





Memorylessness of Exponential (Proof)

If $X \sim Exp(\lambda)$ and $x \geq 0$, then recall

$$P(X > x) = 1 - F_X(x) = 1 - (1 - e^{-\lambda x}) = e^{-\lambda x}$$

$$P(X > s + t \mid X > s) = \frac{Pr(X > s+t, X > s)}{Pr(X > s)}$$

$$= \frac{Pr(X > s+t)}{Pr(X > s)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = \boxed{e^{-\lambda t}} = Pr(X > t)$$

Memorylessness of Exponential (Proof)

If $X \sim Exp(\lambda)$ and $x \geq 0$, then recall

$$P(X > x) = 1 - F_X(x) = 1 - (1 - e^{-\lambda x}) = e^{-\lambda x}$$

$$\begin{aligned} P(X > s + t \mid X > s) &= \frac{P(X > s \mid X > s + t) P(X > s + t)}{P(X > s)} \\ &= \frac{P(X > s + t)}{P(X > s)} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \\ &= e^{-\lambda t} \\ &= P(X > t) \end{aligned}$$

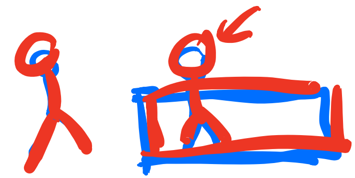
$$\boxed{F_T(x) = 1 - e^{-\lambda x}} \qquad E(X) = \frac{1}{\lambda}$$

EXAMPLE

$$\lambda \quad \frac{\#\text{events}}{\text{min.}}$$

- Time it takes to check someone out at a grocery store is exponential with an $\boxed{\text{expected value of 10 mins.}}$
- Independent for different customers.
- If ~~you are the second person in line,~~ what is the probability that you will have to wait between 10 and 20 mins.

$$\lambda = \frac{1}{10}$$



$$T \sim \exp\left(\frac{1}{10}\right)$$

$$\boxed{\Pr(10 \le T \le 20)}$$

$$= F_T(20) - F_T(10) = \left(1 - e^{-\frac{20}{10}}\right) - \left(1 - e^{-\frac{10}{10}}\right)$$

$$= \int^{20} f_T(x)\,dx \qquad = e^{-1} - e^{-2}$$

10

$\Pr(X \le x) = \Pr(X < x)$

same.

EXAMPLE

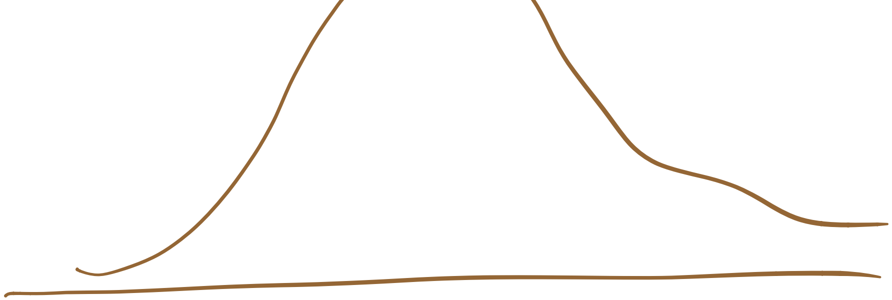
- Time it takes to check someone out at a grocery store is exponential with an expected value of 10 mins.
- Independent for different customers
- If you are the second person in line, what is the probability that you will have to wait between 10 and 20 mins.

$$T \sim exp(10^{-1})$$

$$Pr(10 \le T \le 20) = \boxed{\int_{10}^{20} \frac{1}{10} e^{-\frac{x}{10}} dx}$$

$$\boxed{y = \frac{x}{10} \qquad dy = \frac{1}{10} dx}$$

$$Pr(10 \le T \le 20) = \int_{1}^{2} e^{-y} dy = - e^{-y}\Big|_1^2 = (e^{-1} - e^{-2})$$



4.3 The Normal/Gaussian Random Variable

bell curve.

Agenda

- The Normal/Gaussian RV
- Closure properties of the Normal RV
- The standard normal CDF

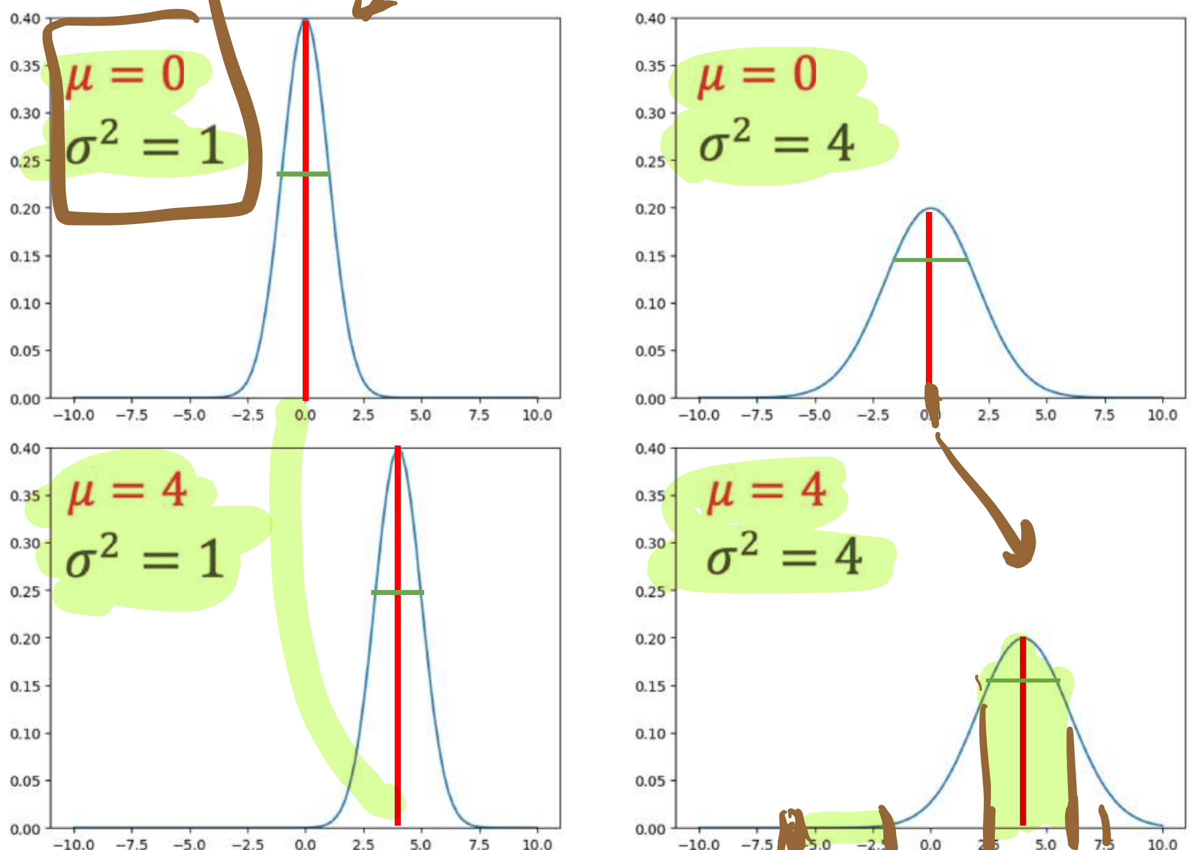
The Normal/Gaussian RV

<u>Normal (Gaussian, "bell curve") Distribution:</u> $X \sim \mathcal{N}(\mu, \sigma^2)$ if and only if X has the following pdf:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$E[X] = \mu \qquad Var(X) = \sigma^2$$

THE NORMAL PDF



$X \sim N(4,4)$

Random Picture

$N(\mu, \sigma^2)$

The Standard Normal CDF Z

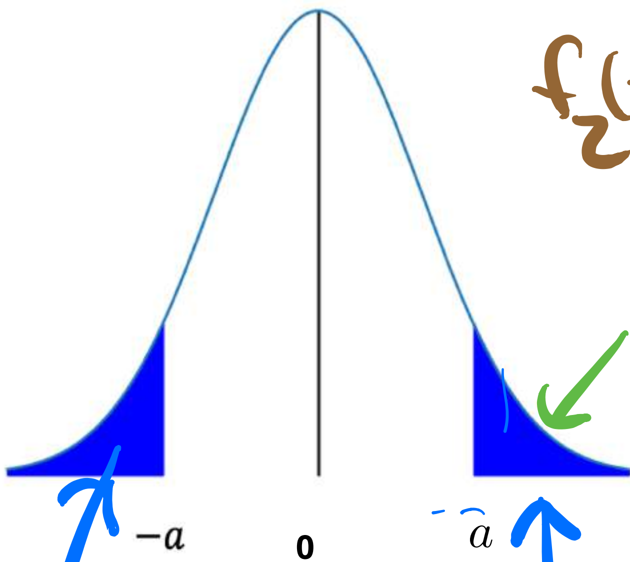
$$\boxed{f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$$

$$f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\boxed{\begin{aligned}\mu &= 0 \\ \sigma^2 &= 1\end{aligned}}$$

$$\boxed{F_Z(x) = \Phi(x)}$$

$$\Pr(Z \le a) = \Phi(a)$$



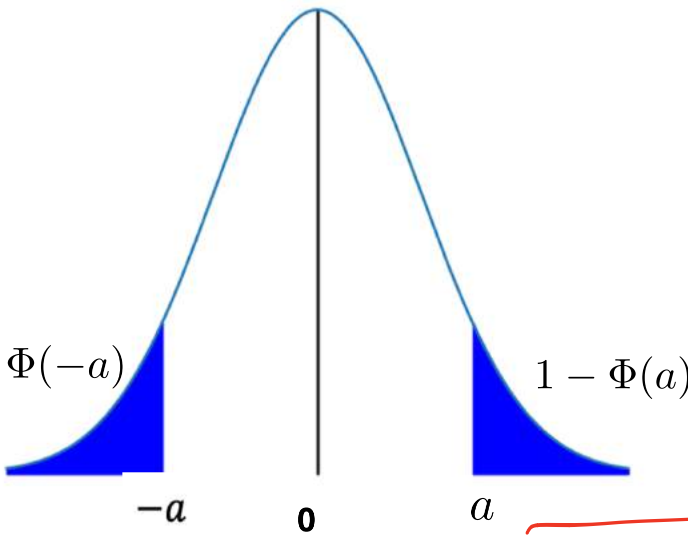
$$\Pr(Z > a) = 1 - \Phi(a)$$

$$\Pr(Z \le -a) = \Phi(-a) = 1 - \Phi(a)$$

The Standard Normal CDF

If $Z \sim \mathcal{N}(0,1)$, we denote the CDF $\Phi(a) = F_Z(a) = P(Z \leq a)$, since it's so commonly used. There is no closed-form formula, so this CDF is stored in a Φ table.

$$\Phi(-a) = 1 - \Phi(a)$$



$$\Phi(1.3) = \int_{-\infty}^{1.3} f_Z(x)\,dx$$

$$\boxed{= \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz}$$

$Z \sim N(0,1)$

The Standard Normal CDF

$$P(Z \leq 1.09) = \Phi(1.09) \approx 0.8621$$

1.61

Φ Table: $\mathbb{P}(Z \leq z)$ when $Z \sim \mathcal{N}(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

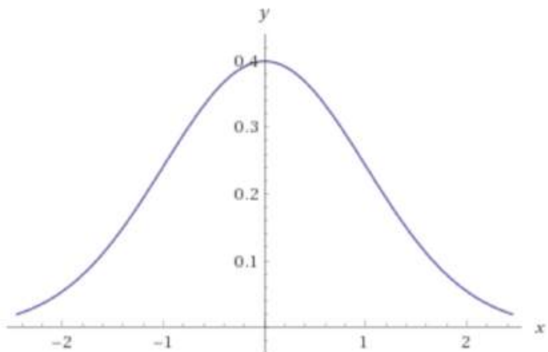
$Pr(Z \leq 0.58)$

a) 0.69146

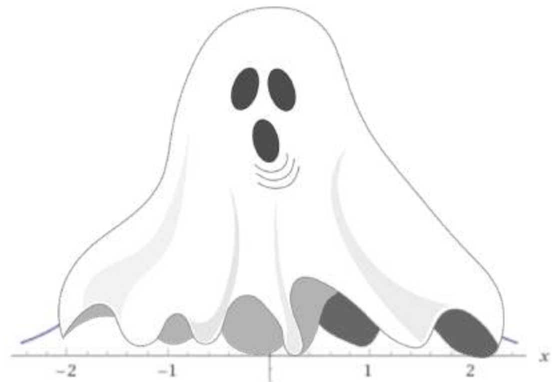
b) 0.8621

c) $\boxed{0.71904}$

d) 0.53188



NORMAL DISTRIBUTION



PARANORMAL DISTRIBUTION