
Anna Karlin
Most Slides by Alex Tsun

Poisson RV Example

Suppose Lookbook gets on average 120 new users per hour, and Quickgram gets 180 new users per hour, independently. What is the probability that, combined, less than 2 users sign up in the next minute?

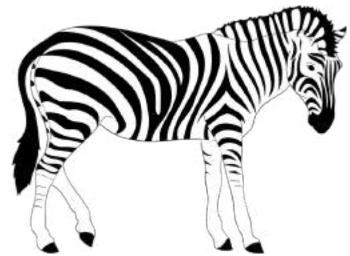
Convert λ's to the same unit of interest. For us, it's a minute.

$$X \sim Poi(2 \text{ users/min}) \qquad Y \sim Poi(3 \text{ users/min})$$

$$Z = X + Y \sim Poi(2+3) = Poi(5)$$

$$P(Z < 2) = p_Z(0) + p_Z(1) = e^{-5}\frac{5^0}{0!} + e^{-5}\frac{5^1}{1!} = 6e^{-5} \approx 0.04$$

The Zoo of Discrete RV's



- The Bernoulli RV
- The Binomial RV
- The Geometric RV
- The Uniform RV
- The Poisson RV

- The Negative Binomial RV
- The Hypergeometric RV

Important Examples:

Uniform(a,b): $P(X = i) = \frac{1}{b-a+1}$ $\mu = \frac{a+b}{2}, \sigma^2 = \frac{(b-a)(b-a+2)}{12}$

Bernoulli(p): $P(X = 1) = p$, $P(X = 0) = 1\text{-}p$ $\mu = p$, $\sigma^2 = p(1\text{-}p)$

Binomial(n,p) $P(X = i) = \binom{n}{i} p^i (1-p)^{n-i}$ $\mu = np$, $\sigma^2 = np(1\text{-}p)$

Poisson(λ): $P(X = i) = e^{-\lambda} \frac{\lambda^i}{i!}$ $\mu = \lambda$, $\sigma^2 = \lambda$

Bin(n,p) $\approx$ Poi(λ) where $\lambda = np$ fixed, $n \to \infty$ (and so $p = \lambda/n \to 0$)

Geometric(p) $P(X = k) = (1\text{-}p)^{k-1}p$ $\mu = 1/p$, $\sigma^2 = (1\text{-}p)/p^2$

$\mu \triangleq E(X)$

$\sigma^2 \triangleq Var(X)$

x

Probability

4.1 Continuous random Variables Basics

Anna Karlin
Most Slides by Alex Tsun

Agenda

- Probability Density Functions (PDFs)
- Cumulative Distribution Functions (CDFs)
- From Discrete to Continuous

Discrete r.v. X with range $\Omega_X = \{0, 1, 2, \ldots\}$

pmf

$p_X(\omega) = \Pr(X = \omega)$

$\forall \omega \quad p_X(\omega) \geq 0$

$\sum_{\omega \in \Omega_X} p_X(\omega) = 1$

CDF

$F_X(\omega) = \Pr(X \leq \omega)$

$\omega_1 < \omega_2$

$\Pr(X \leq \omega_1) \leq \Pr(X \leq \omega_2)$

$F_X(\cdot)$ monotone (weakly) increasing from 0 to 1

$p_X(k) = F_X(k) - F_X(k-1)$	$F_X(\omega) = \sum_{\substack{v \in \Omega_X \\ \text{s.t. } v \leq \omega}} p_X(v)$

given $F_X(\cdot)$, what is $p_X(k)$?

a) $p_X(k) = F_X(k-1)$

b) $p_X(k) = \sum_{\substack{t \in \Omega_X \\ \text{s.t. } t \le k}} F_X(t)$

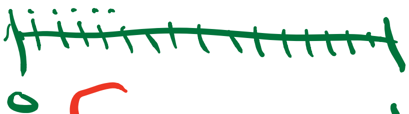
c) $p_X(k) = F_X(k) - F_X(k-1)$ $\Pr(X \le k) - \Pr(X \le k-1)$

d) I don't know

Suppose want to represent cont r.v. uniform random draw from $(0,1]$

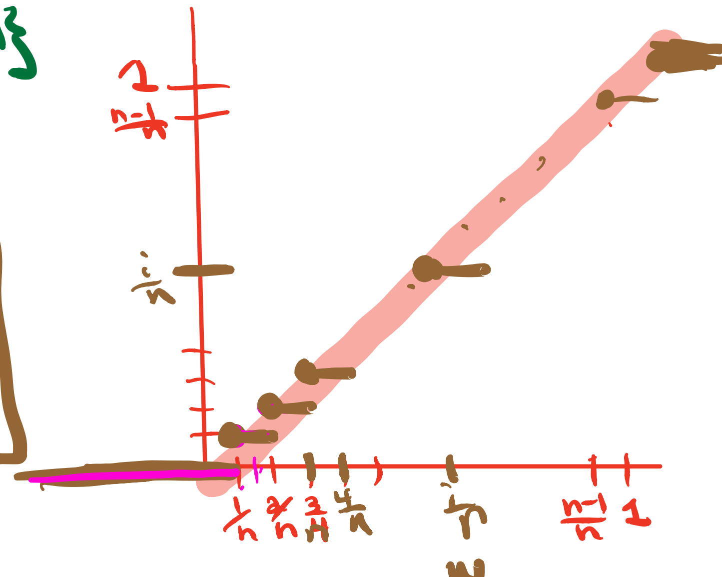
Discrete approx $\Omega_X = \{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$

known CDF discrete distn



$$p_X(w) = \begin{cases} \frac{1}{n} & w \in \{\frac{1}{n}, \frac{2}{n}, \ldots, 1\} \\ 0 & \text{o.w.} \end{cases}$$

$$F_X(w) = \begin{cases} 0 & w < 0 \\ \frac{j}{n} & \frac{j}{n} \le w < \frac{j+1}{n} \quad j = 0, \ldots, n-1 \\ 1 & w \ge 1 \end{cases}$$



let $n \to \infty$

$$F_X(w) = \begin{cases} 0 & w < 0 \\ w & 0 \le w \le 1 \\ 1 & w > 1 \end{cases}$$

as $n \to \infty$ | $p_X(w) = \frac{1}{n} \longrightarrow 0$

p.m.f. no longer makes sense

introduce probability density fn (pdf)
(cont analogue of pmf)

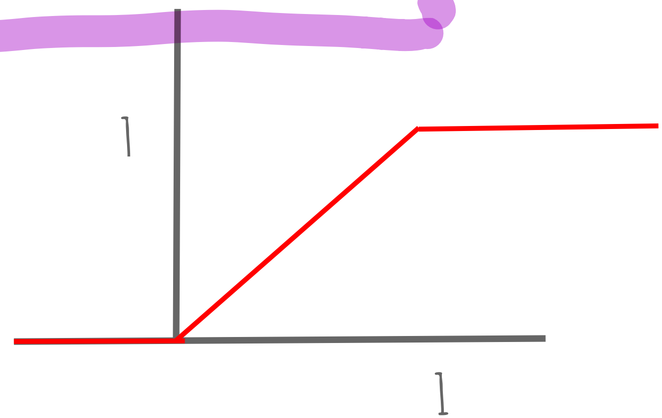
$$p_X(k) = \frac{F_X(k) - F_X(k-1)}{k-(k-1)}$$

$$F_X(w) = \sum_{\substack{v \in \Omega_X \\ \text{s.t. } v \le w}} p_X(v)$$

$$f_X(v) = \frac{d}{dv} F_X(v)$$

CDF INTUITION

$$F_X(w) = P(X \le w)$$

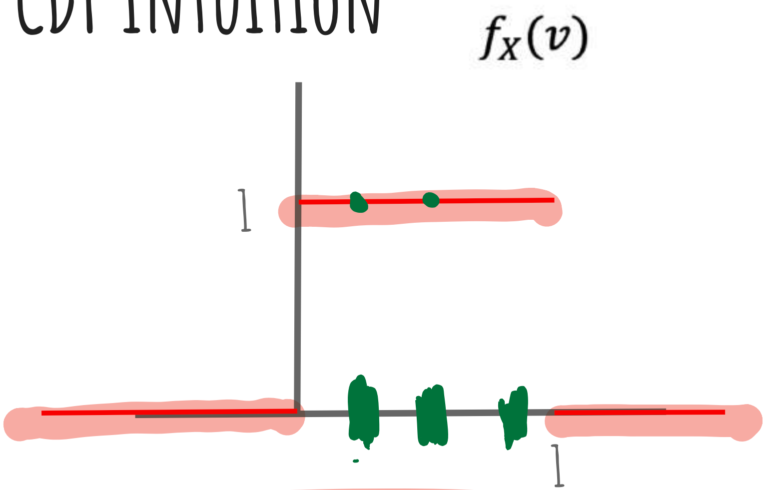


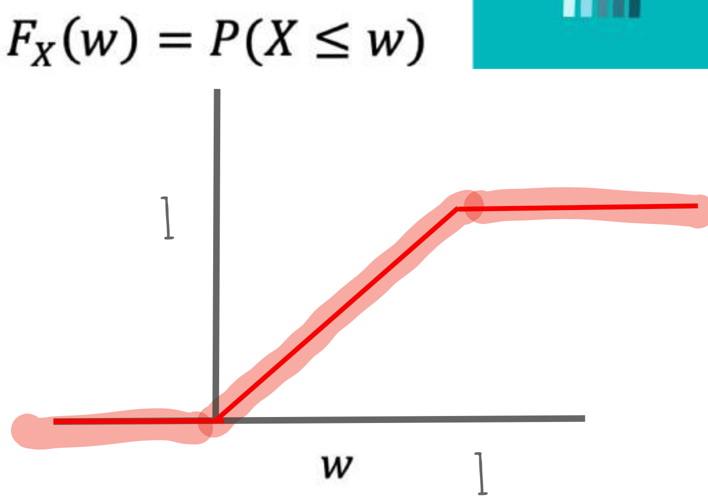
$$F_X(w) = \begin{cases} 0, & w < 0 \\ w, & 0 \le w \le 1 \\ 1, & w > 1 \end{cases}$$

define pdf

$$f_X(w) = \frac{d}{dw} F_X(w)$$

CDF Intuition





$$f_X(v) = \begin{cases} 1, & 0 \leq v \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$F_X(w) = \begin{cases} 0, & w < 0 \\ w, & 0 \leq w \leq 1 \\ 1, & w > 1 \end{cases}$$

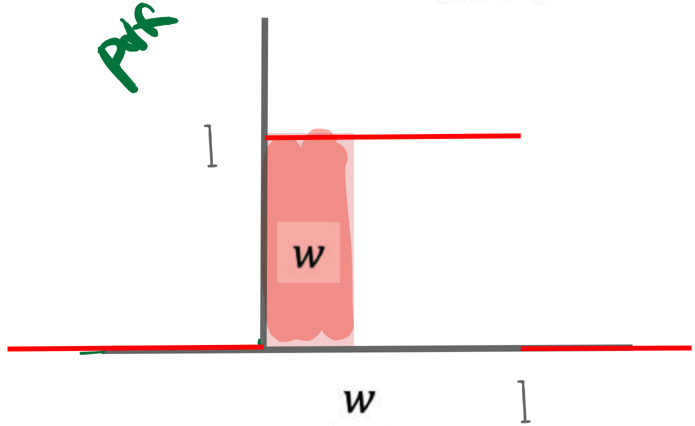
$$f_X(v) = \frac{d}{dv} F_X(v)$$

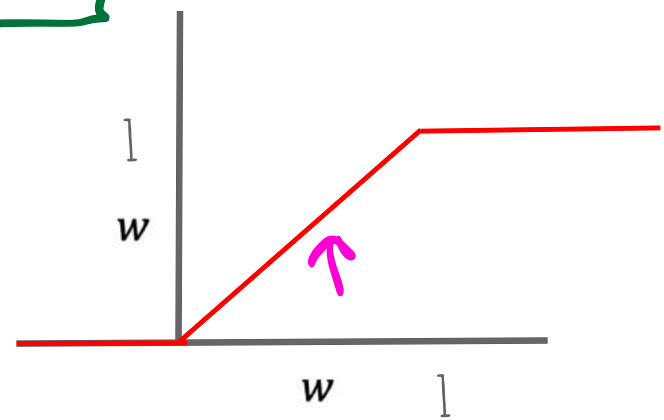
CDF Intuition

pdf

$f_X(v)$

$F_X(w) = P(X \leq w)$





$$f_X(v) = \begin{cases} 1, & 0 \leq v \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$F_X(w) = \begin{cases} 0, & w < 0 \\ w, & 0 \leq w \leq 1 \\ 1, & w > 1 \end{cases}$$

$p_X(x) \geq 0$

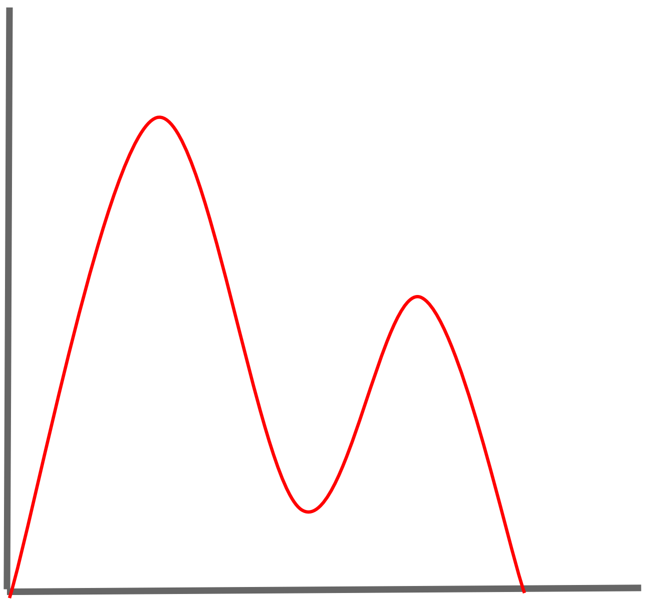
$\sum_{x \in \Omega_X} p_X(x) = 1$

prob density fn.

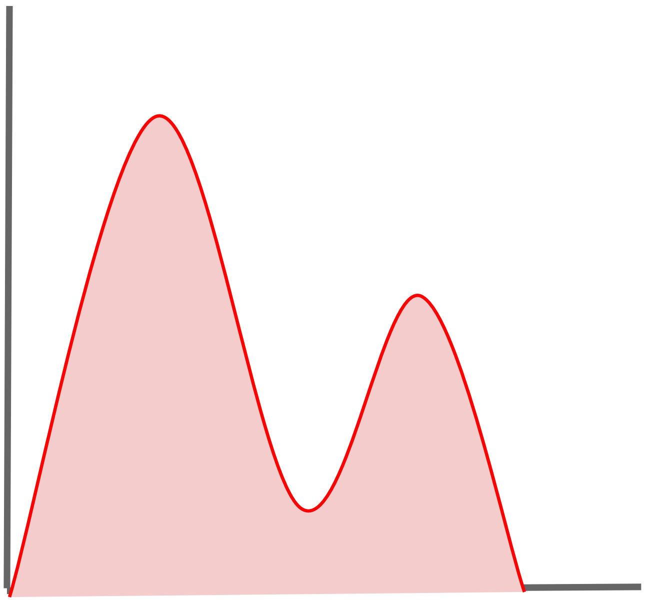
PDF INTUITION

Properties of a pdf

$f_X(z) \geq 0$ for all $z \in \mathbb{R}$



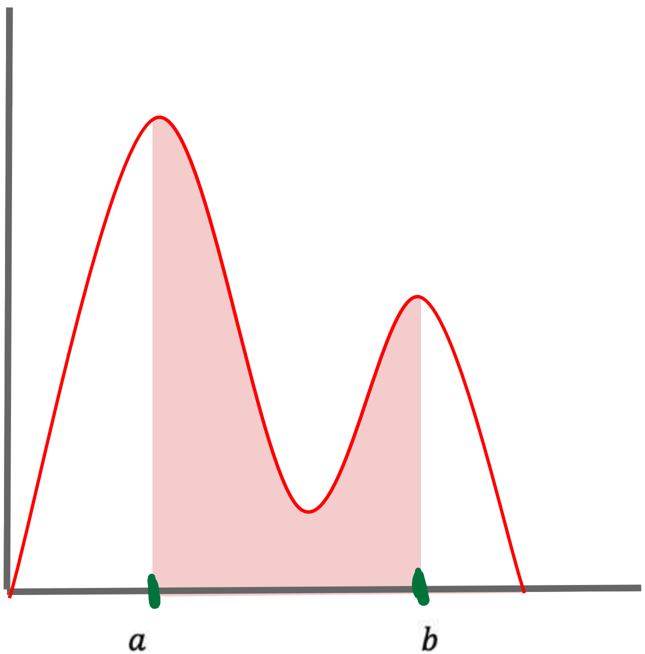
PDF Intuition



$$f_X(z) \geq 0 \text{ for all } z \in \mathbb{R}$$

$$\int_{-\infty}^{\infty} f_X(t)dt = 1$$

PDF Intuition



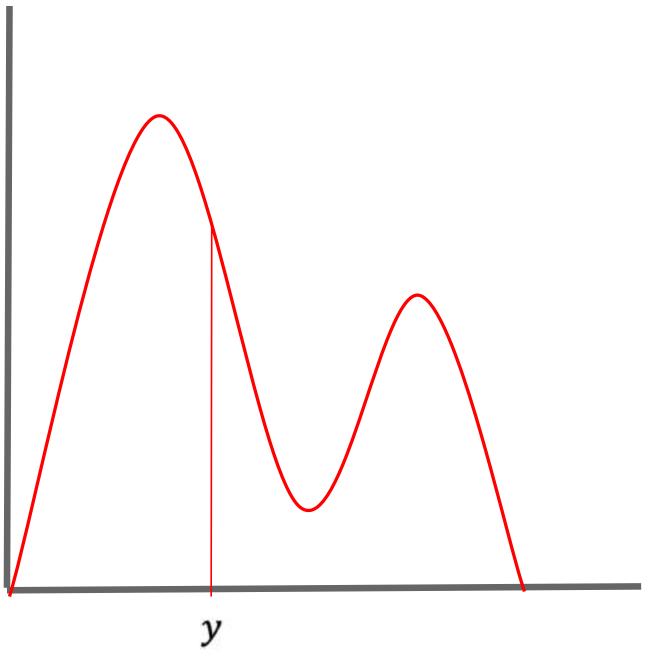
Defn of what pdf must satisfy.

→ $f_X(z) \geq 0$ for all $z \in \mathbb{R}$

→ $$\int_{-\infty}^{\infty} f_X(t)\,dt = 1$$

→ $$P(a \leq X \leq b) = \int_a^b f_X(w)\,dw$$

PDF Intuition



PDF

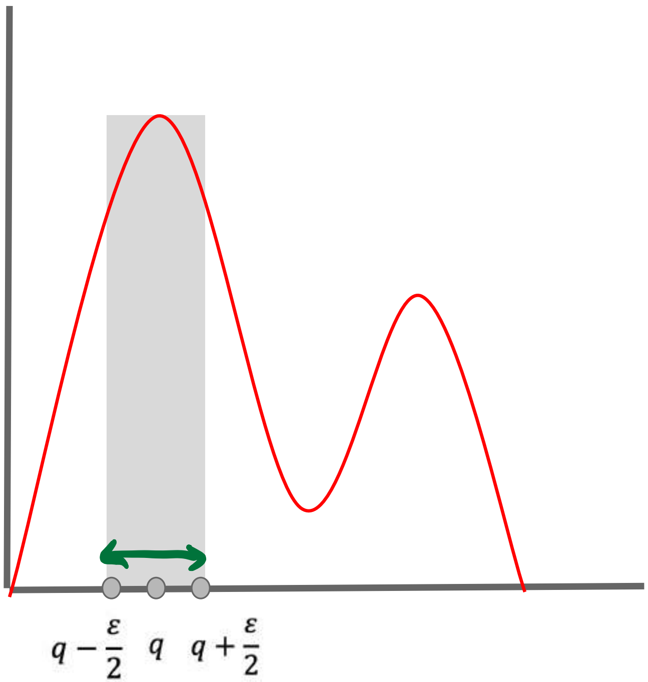
$$f_X(z) \geq 0 \text{ for all } z \in \mathbb{R}$$

$$\int_{-\infty}^{\infty} f_X(t)dt = 1$$

$$P(a \leq X \leq b) = \int_a^b f_X(w)dw$$

$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(w)dw = 0$$

PDF Intuition



PDF

$$f_X(z) \geq 0 \text{ for all } z \in \mathbb{R}$$

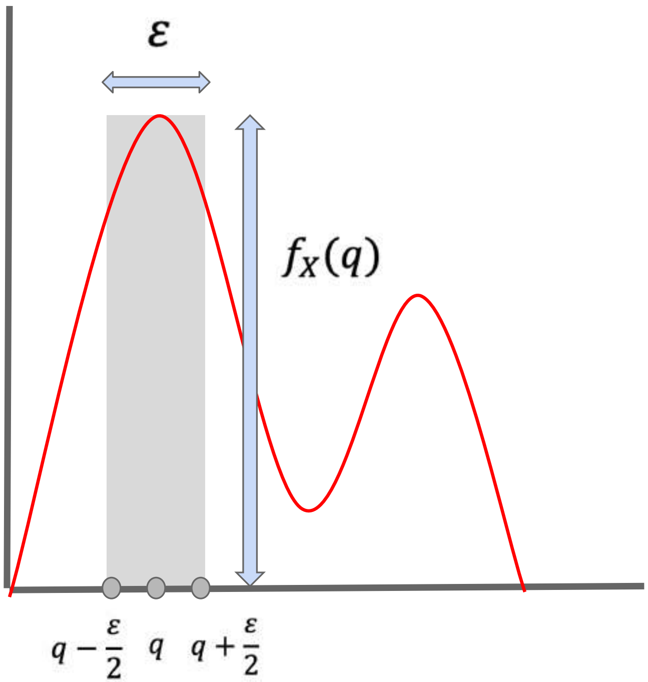
$$\int_{-\infty}^{\infty} f_X(t)dt = 1$$

$$P(a \leq X \leq b) = \int_a^b f_X(w)dw$$

$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(w)dw = 0$$

$$P(X \approx q) \approx P\left(q - \frac{\varepsilon}{2} \leq X \leq q + \frac{\varepsilon}{2}\right)$$

PDF Intuition



$$f_X(z) \geq 0 \text{ for all } z \in \mathbb{R}$$

$$\int_{-\infty}^{\infty} f_X(t)dt = 1$$

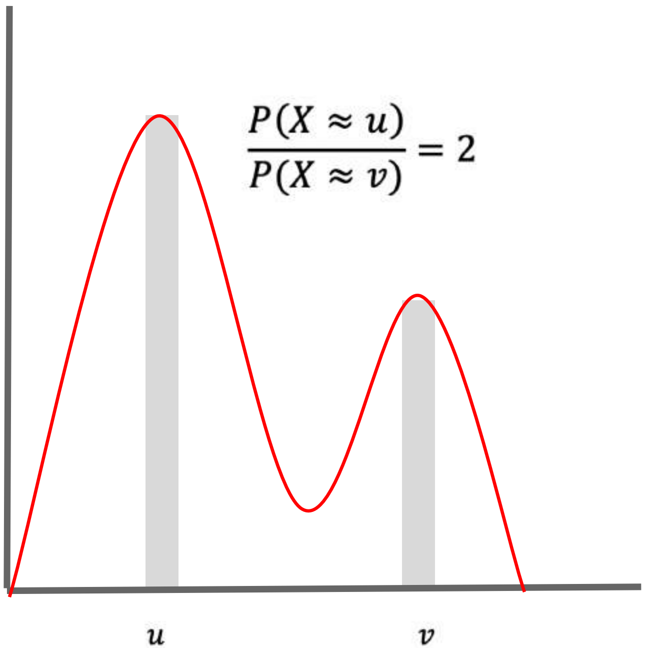
$$P(a \leq X \leq b) = \int_a^b f_X(w)dw$$

$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(w)dw = 0$$

$$P(X \approx q) \approx P\left(q - \frac{\varepsilon}{2} \leq X \leq q + \frac{\varepsilon}{2}\right) \approx \varepsilon f_X(q)$$

$\Pr(X \approx q)$ proportional $f_X(q)$

PDF Intuition



$$\frac{P(X \approx u)}{P(X \approx v)} = 2$$

$$f_X(z) \geq 0 \text{ for all } z \in \mathbb{R}$$

$$\int_{-\infty}^{\infty} f_X(t)dt = 1$$

$$P(a \leq X \leq b) = \int_a^b f_X(w)dw$$

$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(w)dw = 0$$

$$P(X \approx q) \approx P\left(q - \frac{\varepsilon}{2} \leq X \leq q + \frac{\varepsilon}{2}\right) \approx \varepsilon f_X(q)$$

$$\frac{P(X \approx u)}{P(X \approx v)} = \frac{\varepsilon f_X(u)}{\varepsilon f_X(v)} = \frac{f_X(u)}{f_X(v)}$$

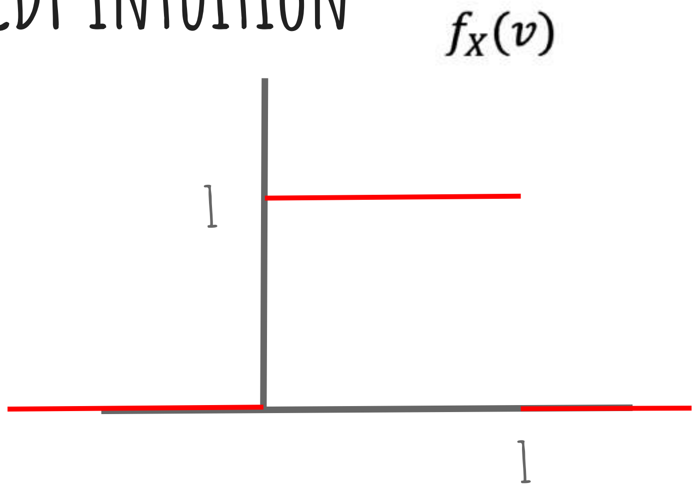
Probability Density Functions (PDFs)

Probability Density Function (PDF): Let X be a continuous rv (one whose range is typically an interval or union of intervals). The probability density function (PDF) of X is the function $f_X: \mathbb{R} \to \mathbb{R}$ such that

- $f_X(z) \geq 0$ for all $z \in \mathbb{R}$.
- $\int_{-\infty}^{\infty} f_X(t)dt = 1$.
- $P(a \leq X \leq b) = \int_a^b f_X(w)dw$. ⇐ $Pr(X \leq b) - Pr(X < a)$
  $F_X(b) - F_X(a)$
- $P(X = y) = 0$ for any $y \in \mathbb{R}$.
- The probability that X is close to q is proportional to $f_X(q)$: $P(X \approx q) \approx P\left(q - \frac{\varepsilon}{2} \leq X \leq q + \frac{\varepsilon}{2}\right) \approx \varepsilon f_X(q)$.
- Ratios of probabilities of being near points are maintained: $\frac{P(X \approx u)}{P(X \approx v)} = \frac{\varepsilon f_X(u)}{\varepsilon f_X(v)} = \frac{f_X(u)}{f_X(v)}$.

Random Picture

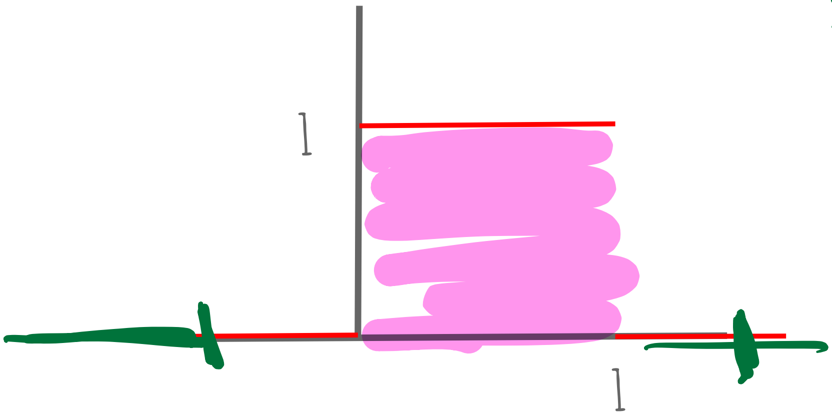
CDF Intuition



$$f_X(v) = \begin{cases} 1, & 0 \leq v \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

CDF Intuition

$f_X(v)$



$$F_X(w) = P(X \le w) = \int_{-\infty}^{w} f_X(v)\,dv$$

$$f_X(v) = \begin{cases} 1, & 0 \le v \le 1 \\ 0, & \text{otherwise} \end{cases}$$

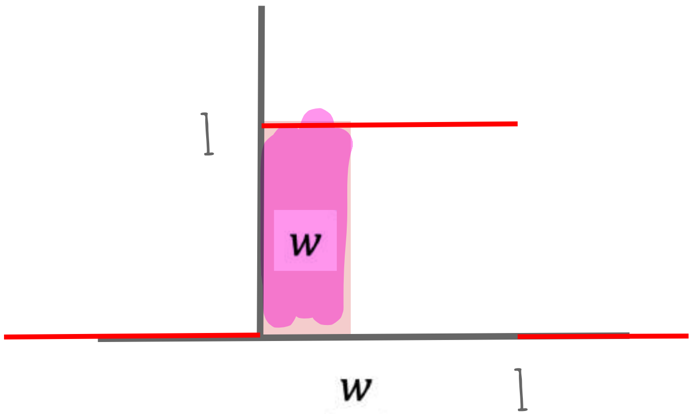
$$F_X(w) = \begin{cases} 0 & w < 0 \\ & 0 \le w \le 1 \\ 1 & w > 1 \end{cases}$$

$0 \le w \le 1$

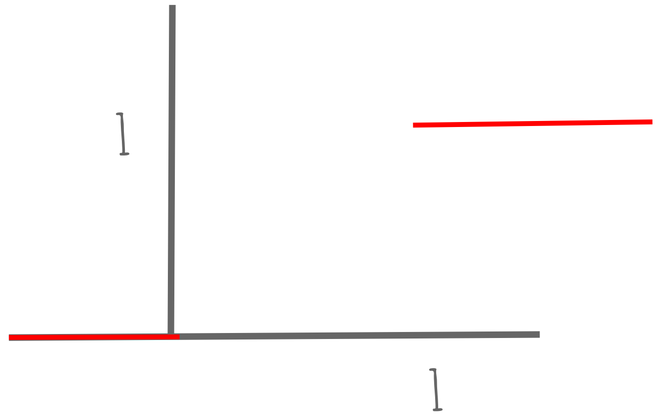
$$= \int_{-\infty}^{w} f_X(v)\,dv = \int_{0}^{w} 1\,dv = \underline{\quad}$$

CDF Intuition

$f_X(v)$



$F_X(w) = P(X \le w)$

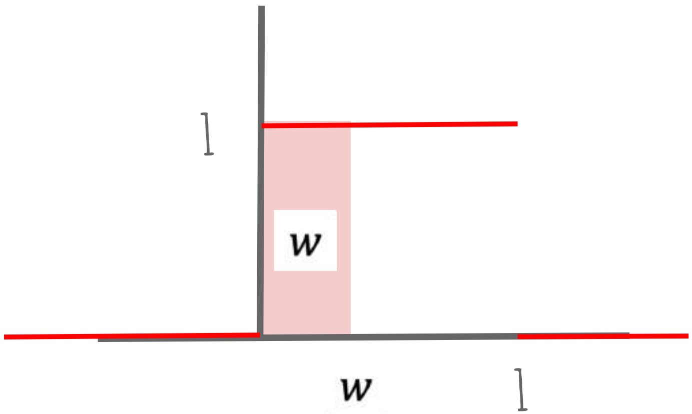


$$f_X(v) = \begin{cases} 1, & 0 \le v \le 1 \\ 0, & \text{otherwise} \end{cases}$$

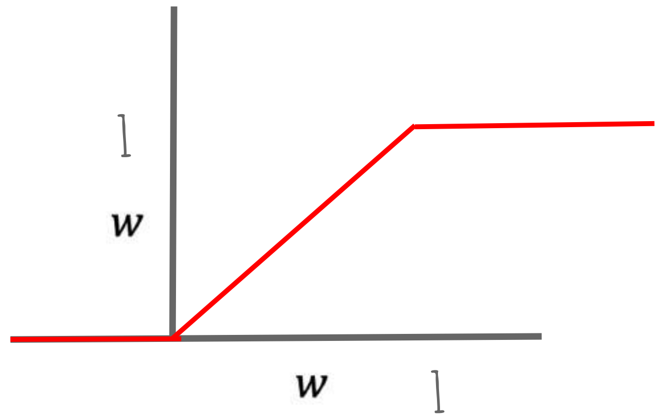
$$F_X(w) = \begin{cases} 0, & w < 0 \\ w & 0 \le w \le 1 \\ 1, & w > 1 \end{cases}$$

CDF Intuition

$f_X(v)$



$F_X(w) = P(X \leq w)$



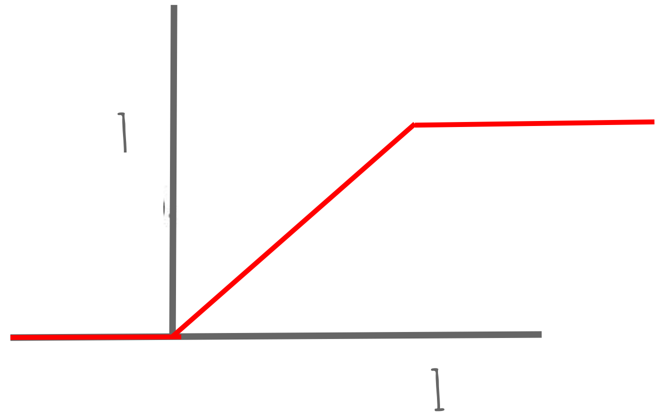
$$f_X(v) = \begin{cases} 1, & 0 \leq v \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$F_X(w) = \begin{cases} 0, & w < 0 \\ w, & 0 \leq w \leq 1 \\ 1, & w > 1 \end{cases}$$

CDF Intuition

$F_X(t) = P(X \le t) = \int_{-\infty}^{t} f_X(w)dw$ for all $t \in \mathbb{R}$.

$$F_X(w) = P(X \le w)$$



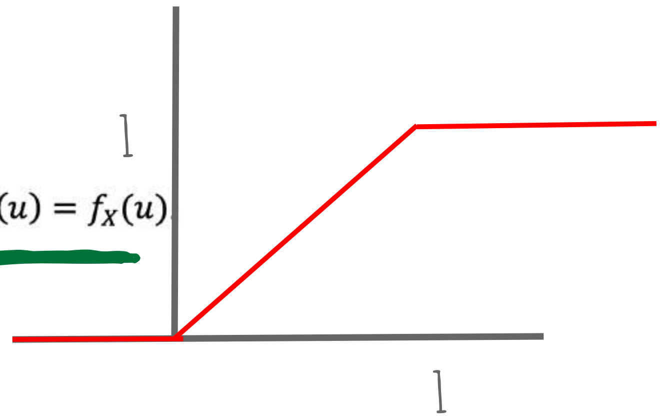
$$F_X(w) = \begin{cases} 0, & w < 0 \\ w, & 0 \le w \le 1 \\ 1, & w > 1 \end{cases}$$

CDF Intuition

$F_X(t) = P(X \leq t) = \int_{-\infty}^{t} f_X(w)dw$ for all $t \in \mathbb{R}$.

Hence, by the Fundamental Theorem of Calculus, $\frac{d}{du}F_X(u) = f_X(u)$

$$F_X(w) = P(X \leq w)$$



$$F_X(w) = \begin{cases} 0, & w < 0 \\ w, & 0 \leq w \leq 1 \\ 1, & w > 1 \end{cases}$$

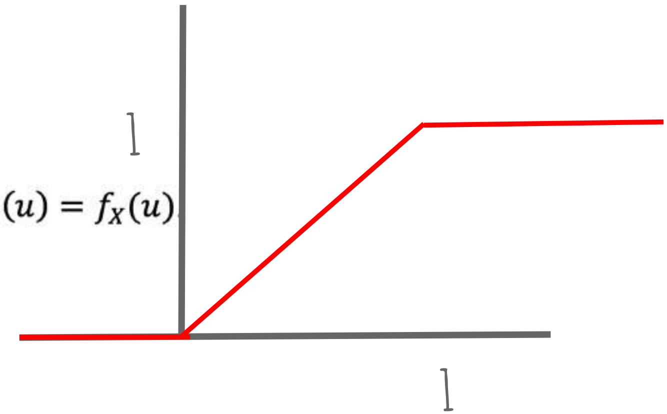
CDF Intuition

$$F_X(w) = P(X \leq w)$$

$F_X(t) = P(X \leq t) = \int_{-\infty}^{t} f_X(w)dw$ for all $t \in \mathbb{R}$.

Hence, by the Fundamental Theorem of Calculus, $\frac{d}{du}F_X(u) = f_X(u)$.

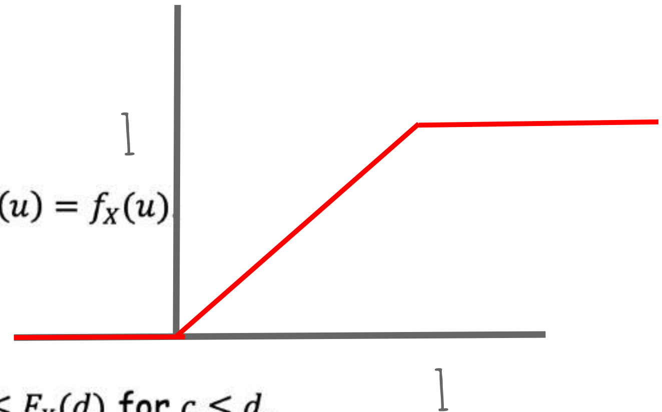
$P(a \leq X \leq b) = F_X(b) - F_X(a)$.



$$F_X(w) = \begin{cases} 0, & w < 0 \\ w, & 0 \leq w \leq 1 \\ 1, & w > 1 \end{cases}$$

CDF Intuition

$$F_X(w) = P(X \le w)$$



$$F_X(w) = \begin{cases} 0, & w < 0 \\ w, & 0 \le w \le 1 \\ 1, & w > 1 \end{cases}$$

$F_X(t) = P(X \le t) = \int_{-\infty}^{t} f_X(w)\,dw$ for all $t \in \mathbb{R}$.

Hence, by the Fundamental Theorem of Calculus, $\frac{d}{du} F_X(u) = f_X(u)$.

$P(a \le X \le b) = F_X(b) - F_X(a)$.

F_X is monotone increasing, since $f_X \ge 0$. That is, $F_X(c) \le F_X(d)$ for $c \le d$.

CDF Intuition

$$F_X(w) = P(X \leq w)$$

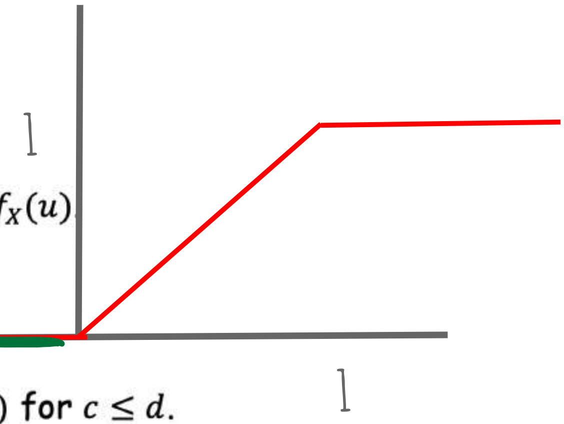
$F_X(t) = P(X \leq t) = \int_{-\infty}^{t} f_X(w)dw$ for all $t \in \mathbb{R}$.

Hence, by the Fundamental Theorem of Calculus, $\frac{d}{du}F_X(u) = f_X(u)$.

$P(a \leq X \leq b) = F_X(b) - F_X(a)$.

F_X is monotone increasing, since $f_X \geq 0$. That is, $F_X(c) \leq F_X(d)$ for $c \leq d$.

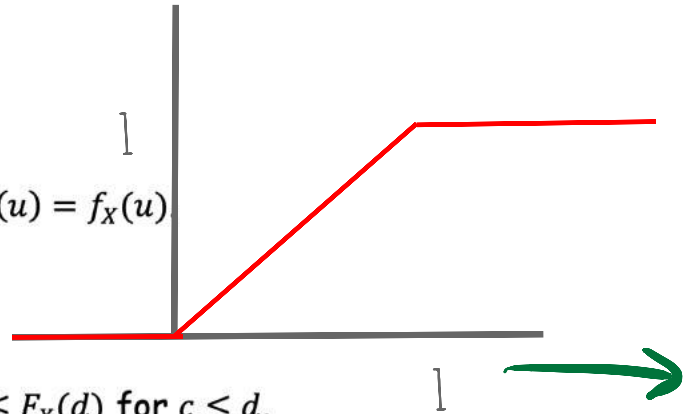
$\lim_{v \to -\infty} F_X(v) = P(X \leq -\infty) = 0$.



$$F_X(w) = \begin{cases} 0, & w < 0 \\ w, & 0 \leq w \leq 1 \\ 1, & w > 1 \end{cases}$$

CDF Intuition

$$F_X(w) = P(X \leq w)$$



$F_X(t) = P(X \leq t) = \int_{-\infty}^{t} f_X(w)dw$ for all $t \in \mathbb{R}$.

Hence, by the Fundamental Theorem of Calculus, $\frac{d}{du}F_X(u) = f_X(u)$.

$P(a \leq X \leq b) = F_X(b) - F_X(a)$.

F_X is monotone increasing, since $f_X \geq 0$. That is, $F_X(c) \leq F_X(d)$ for $c \leq d$.

$\lim_{v \to -\infty} F_X(v) = P(X \leq -\infty) = 0$.

$\lim_{v \to +\infty} F_X(v) = P(X \leq +\infty) = 1$.

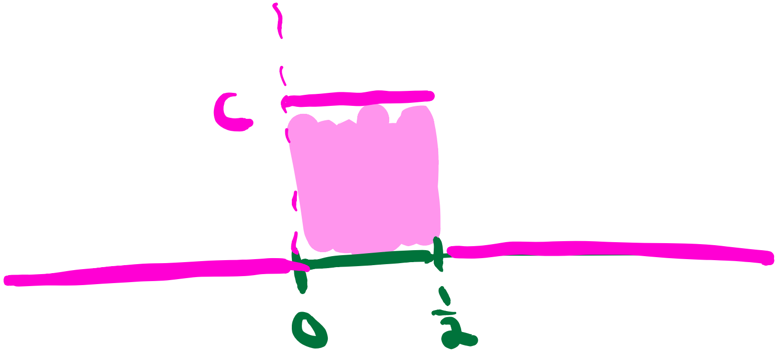
$$F_X(w) = \begin{cases} 0, & w < 0 \\ w, & 0 \leq w \leq 1 \\ 1, & w > 1 \end{cases}$$

Cumulative Distribution Functions (CDFs)

Cumulative Distribution Function (CDF): Let X be a continuous rv (one whose range is typically an interval or union of intervals). The cumulative distribution function (CDF) of X is the function $F_X: \mathbb{R} \to \mathbb{R}$ such that

- $F_X(t) = P(X \leq t) = \int_{-\infty}^{t} f_X(w)dw$ for all $t \in \mathbb{R}$.
- Hence, by the Fundamental Theorem of Calculus, $\frac{d}{du}F_X(u) = f_X(u)$.
- $P(a \leq X \leq b) = F_X(b) - F_X(a)$.
- F_X is monotone increasing, since $f_X \geq 0$. That is, $F_X(c) \leq F_X(d)$ for $c \leq d$.
- $\lim_{v \to -\infty} F_X(v) = P(X \leq -\infty) = 0$.
- $\lim_{v \to +\infty} F_X(v) = P(X \leq +\infty) = 1$.

Model X uniform on $(0, \frac{1}{2})$



What is c?

a) 1

b) $\frac{1}{2}$

c) 2

d) I don't know

$$f_X(v) = \begin{cases} 0 & v < 0 \\ c & 0 \le v \le \frac{1}{2} \\ 0 & v > \frac{1}{2} \end{cases}$$

$$\int_{-\infty}^{\infty} f_X(v)\,dv = 1$$

$$\Downarrow$$

$$\int_0^{1/2} c\,dv = cv\Big|_0^{\frac{1}{2}} = \frac{c}{2} = 1$$

$$\frac{c}{2} = 1 \Rightarrow c = 2$$

$$F_X(v) = ?$$

a) $= \begin{cases} 0 & v < 0 \\ 2v & v \in [0, \frac{1}{2}] \\ 1 & v > \frac{1}{2} \end{cases}$

b) $= \begin{cases} 0 & v < 0 \\ \frac{v}{2} & v \in [0, \frac{1}{2}] \\ 1 & v > \frac{1}{2} \end{cases}$

c) $= \begin{cases} 0 & v < 0 \\ v & v \in [0, \frac{1}{2}] \\ 1 & v > \frac{1}{2} \end{cases}$

d) I don't know

$f_X(x) =$

$F_X(x) \Rightarrow$

$\int_{-\infty}^{\infty} f_X(x) = 1$

From Discrete to Continuous

$f_X(x) = \frac{d}{dx} F_X(x)$

	Discrete	Continuous
PMF/PDF	$p_X(x) = P(X = x)$	$f_X(x) \neq P(X = x) = 0$
CDF	$F_X(x) = \sum_{t \leq x} p_X(t)$	$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$
Normalization	$\sum_x p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$
Expectation	$\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$

$E(X) = \sum_{x \in \Omega_X} x \, p_X(x)$

$E(X) = \int_{-\infty}^{\infty} x \, f_X(x)\, dx$

Probability

4.2 Zoo of Continuous RVs

Agenda

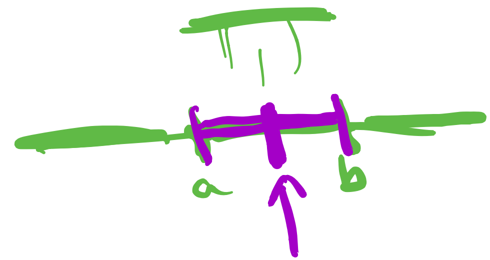
- The (Continuous) Uniform RV
- The Exponential RV
- Memorylessness

The (Continuous) Uniform RV

Uniform (Continuous) RV: $X \sim Unif(a, b)$ where $a < b$ are real numbers, if and only if X has the following pdf:

pdf.

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & x \in [a, b] \\ 0, & \text{otherwise} \end{cases}$$



X is equally likely to take on any value in $[a, b]$.

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_a^b x \cdot \frac{1}{(b-a)}\,dx = \frac{1}{(b-a)} \frac{x^2}{2} \Big|_a^b$$

$$= \frac{b^2 - a^2}{2(b-a)}$$

$$= \frac{(b-a)(b+a)}{2(b-a)}$$

$$= \frac{b+a}{2}$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_a^b x^2 \cdot \frac{1}{(b-a)}\,dx$$

The Uniform (Continuous) RV

Uniform (Continuous) RV: $X \sim Unif(a, b)$ where $a < b$ are real numbers, if and only if X has the following pdf:

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & x \in [a, b] \\ 0, & \text{otherwise} \end{cases}$$

X is equally likely to take on any value in $[a, b]$.

$$E[X] = \frac{a+b}{2} \qquad Var(X) = \frac{(b-a)^2}{12}$$

The cdf is

$$F_X(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \le x \le b \\ 1, & x > b \end{cases}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$$