Probability

3.2 Discrete Random Variables Basics

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Agenda

- Recap on rvs
- Expectation
- Linearity of Expectation (LoE)
- Law of the Unconscious Statistician (LOTUS)

Random Variable

Suppose we conduct an experiment with sample space Ω. A **random variable (rv)** is a numeric function of the outcome, $X: \Omega \to \mathbb{R}$. That is, it maps outcomes $\omega \in \Omega$ to numbers, $\omega \mapsto X(\omega)$.

The set of possible values X can take on is its **range/support**, denoted Ω_X.

If Ω_X is finite or countably infinite (typically integers or a subset), X is a **discrete** random variable (drv). Else if Ω_X is uncountably large (the size of real numbers), X is **continuous** random variable.

Probability Mass Function (PMF)

The **probability mass function (pmf)** of a discrete random variable X assigns probabilities to the possible values of the random variable. That is, $p_X\colon \Omega_X \to [0,1]$ where

$$p_X(k) = P(X = k)$$

Note that $\{X = a\}$ for $a \in \Omega_X$ form a partition of Ω, since each outcome $\omega \in \Omega$ is mapped to exactly one number. Hence,

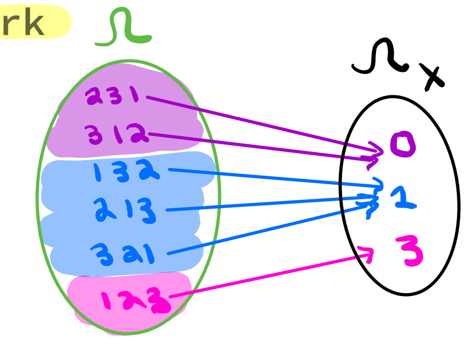
$$\sum_{z \in \Omega_X} p_X(z) = 1$$

Homeworks of 3 students returned randomly

$w \in \Omega \to X(w)$

- Each permutation equally likely
- X: # people who get their own homework

Prob	Outcome w	X(w)
1/6	1 2 3	3
1/6	1 3 2	1
1/6	2 1 3	1
1/6	2 3 1	0
1/6	3 1 2	0
1/6	3 2 1	1



pmf

$$p_X(k) = \begin{cases} \frac{1}{3} & k=0 \\ \frac{1}{2} & k=1 \\ \frac{1}{6} & k=3 \\ 0 & \text{otherwise} \end{cases}$$

CDF

$$F_X(x) = \begin{cases} 0 & x<0 \\ \frac{1}{3} & 0 \le x < 1 \\ \frac{5}{6} & 1 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

$F_X(x) = Pr(X \le x)$

$$\sum_{k \in \Omega_X} p_X(k) = 1$$

Expectation

The **<u>expectation/expected value/average</u>** of a discrete random variable X is

$$E[X] = \sum_{\omega \in \Omega} X(\omega) P(\omega)$$

Or equivalently,

$$E[X] = \sum_{k \in \Omega_X} k \cdot p_X(k)$$

The interpretation is that we take an average of the values in Ω_X, but weighted by their probabilities.

Homeworks of 3 students returned randomly

- Each permutation equally likely
- X: # people who get their own homework
- What is E(X)?

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pmf

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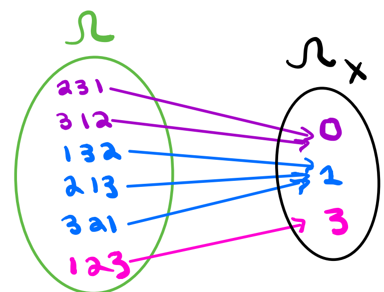


$$\boxed{E(X) = \sum_{k \in \Omega_X} k \, p_X(k) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{2} + 3 \cdot \frac{1}{6}} = 1$$

$$= \sum_{\omega \in \Omega} X(\omega) P(\omega)$$

Repeated coin flipping

Flip a biased coin with probability p of coming up Heads n times. Each flip independent of all others.

X is number of Heads.

What is E(X)?

$$E(X) = \sum_{k=0}^{n} k \underbrace{P(X=k)}_{p_X(k)}$$

$P(X=k) = p_X(k) = ?$

a) $\binom{n}{k} p^k$

b) $p^k (1-p)^{n-k}$

c) $\binom{n}{k} (1-p)^{n-k}$

d) $\boxed{\binom{n}{k} p^k (1-p)^{n-k}}$

$n=5$

$$Pr(X=2) = \binom{5}{2} \underbrace{p^2(1-p)^3}$$

$\Rightarrow \boxed{HHTTT}$

$p^2(1-p)^3$

$\Rightarrow HTTHT$

Repeated coin flipping

Flip a biased coin with probability p of coming up Heads n times.

X is number of Heads.

What is E(X)?

$$\boxed{E[X] = \sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i}}$$

$$= \sum_{i=1}^{n} i \binom{n}{i} p^i (1-p)^{n-i}$$

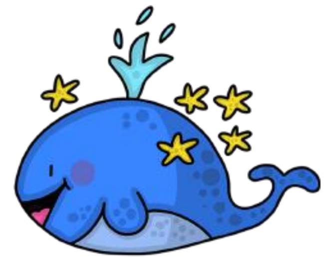
$$= \sum_{i=1}^{n} n \binom{n-1}{i-1} p^i (1-p)^{n-i}$$

$$= np \sum_{i=1}^{n} \binom{n-1}{i-1} p^{i-1} (1-p)^{n-i}$$

$$= np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-1-j}$$

$$= np(p + (1-p))^{n-1} = np$$

Linearity of Expectation (Idea)



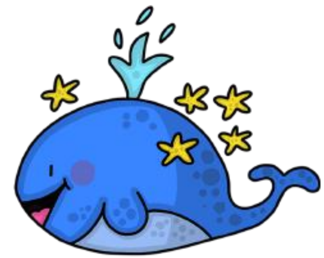
Let's say you and your friend sell fish for a living.

- Every day you catch X fish, with $E[X] = 3$.
- Every day your friend catches Y fish, with $E[Y] = 7$.

How many fish do the two of you bring in $(Z = X + Y)$ on an average day?

$E[Z] = E[X + Y] = E(X) + E(Y) = 3 + 7 = 10$

Linearity of Expectation (Idea)



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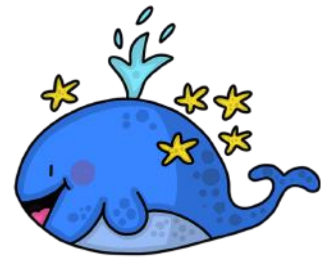

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$$E[Z] = E[X + Y] = E[X] + E[Y] = 3 + 7 = 10$$

You can sell each fish for \$5 at a store, but you need to pay \$20 in rent. How much profit do you expect to make? $E[5Z - 20] = 5E(Z) - 20$

$$= 5 \cdot 10 - 20 = 30$$

Linearity of Expectation (Idea)

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Linearity of Expectation (LoE)

Linearity of Expectation: Let Ω be the sample space of an experiment, $X, Y: \Omega \to \mathbb{R}$ be (possibly "dependent") random variables both defined on Ω, and $a, b, c \in \mathbb{R}$ be scalars. Then,

$$E[X + Y] = E[X] + E[Y]$$

and

$$E[aX + b] = aE[X] + b$$

Combining them gives,

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

Linearity of Expectation (Proof)

$$E[X] + E[Y] = \sum_{\omega \in \Omega} X(\omega)P(\omega) + \sum_{\omega \in \Omega} Y(\omega)P(\omega)$$

$$= \sum_{\omega \in \Omega} \big(X(\omega) + Y(\omega)\big)P(\omega)$$

$$= \sum_{\omega \in \Omega} (X + Y)(\omega)\, P(\omega)$$

$$= E[X + Y]$$

Corollary: linearity for sum of lots of r.v.s

$$E(X_1 + X_2 + \ldots + X_n) = E(X_1) + E(X_2) + \ldots + E(X_n)$$

Proof by induction!

Homeworks of students returned randomly

- Each permutation equally likely
- X: # people who get their own homework
- What is E(X) when there are n students?

Prob	Outcome w	X	X_1	X_2	X_3
1/6	1 2 3	3	1	1	1
1/6	1 3 2	1	1	0	0
1/6	2 1 3	1	0	0	1
1/6	2 3 1	0	0	0	0
1/6	3 1 2	0	0	0	0
1/6	3 2 1	1	0	1	0

$$E(X) = \sum_{k=0}^{n} k\,\underbrace{P(X=k)}$$

$$X_i = \begin{cases} 1 & \text{if student } i \text{ gets hw back} \\ 0 & \text{otherwise} \end{cases}$$

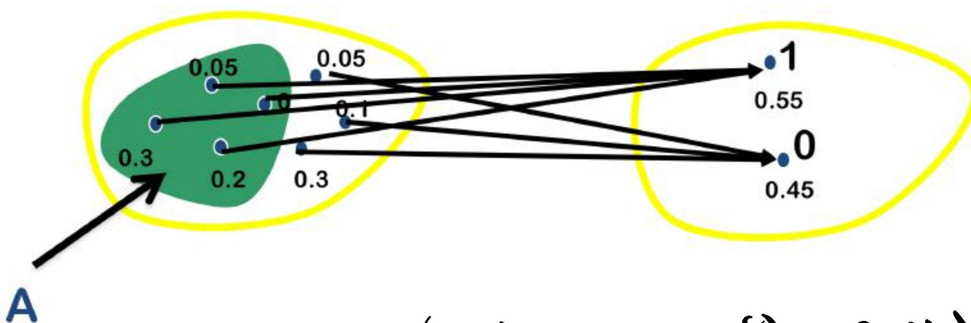
$$X = X_1 + X_2 + X_3$$

$$E(X) = E(X_1 + X_2 + X_3)$$
$$= E(X_1) + E(X_2) + E(X_3)$$

$$E(X_i) = 0 \cdot Pr(X_i = 0) + 1\,Pr(X_i = 1)$$
$$= Pr(X_i = 1) = \frac{1}{3}$$

Indicator random variable

- For any event A, can define the indicator random variable for A



$$E(X_A) = 0 \cdot Pr(A^c) + 1 \cdot Pr(A)$$
$$= Pr(A)$$

X_i: A: event that student i got own HW back.

Computing complicated expectations

- Often boils down to finding the right way to decompose the random variable into simple random variables (often indicator random variables) and then applying linearity of expectation.

Repeated coin flipping

Flip a biased coin with probability p of coming up Heads n times.

X is number of Heads.

What is E(X)?

$$\begin{aligned}
E[X] &= \sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i} \\
&= \sum_{i=1}^{n} i \binom{n}{i} p^i (1-p)^{n-i} \\
&= \sum_{i=1}^{n} n \binom{n-1}{i-1} p^i (1-p)^{n-i} \\
&= np \sum_{i=1}^{n} \binom{n-1}{i-1} p^{i-1} (1-p)^{n-i} \\
&= np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} \\
&= np(p + (1-p))^{n-1} = np
\end{aligned}$$

REPEATED COIN FLIPPING n times independently

Flip a biased coin with probability p of coming up Heads n times.

X is number of Heads.

What is E(X)?

$X = X_1 + X_2 + \dots + X_n$

$X_i = \begin{cases} 1 & \text{if } i^{th} \text{ flip is Heads} \\ 0 & \text{otherwise.} \end{cases}$

$E(X_i) = Pr(i^{th} \text{ flip is } H) = p$

$E(X) = E(X_1 + X_2 + \dots + X_n) = \underbrace{E(X_1)}_{p} + \underbrace{E(X_2)}_{p} + \dots + \underbrace{E(X_n)}_{p}$

$= np$

Pairs with same birthday

$|\Omega| = 365^m$

uniform sample space

indep. each student has random bday

- In a class of m students, on average how many pairs of people have the same birthday?

X: # pairs of people with same bday.

$E(X) = \sum_{k=0}^{\binom{m}{2}} k \Pr(X=k)$

1, 2, ..., m names of people.

$X_{ij} = \begin{cases} 1 & \text{student } i \text{ \& student } j \text{ have same bday} \\ 0 & \text{o.w.} \end{cases}$ $i<j$ are 2 students in class.

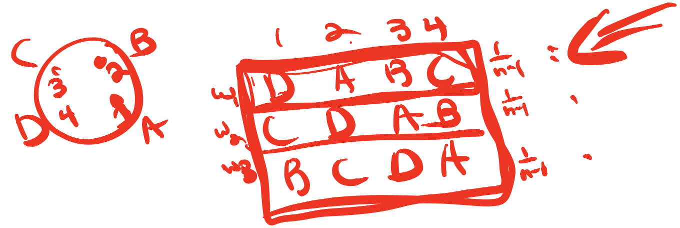
$\boxed{E(X)} = E\left(\sum_{\text{pairs }(i,j)} X_{ij}\right) = \sum_{\text{pairs }(i,j)} \underline{E(X_{ij})} = \sum_{\text{pairs }(i,j)} \frac{1}{365} = \binom{m}{2}\frac{1}{365}$

$\boxed{E(X_{ij})} = \Pr(\text{student } i \text{ \& student } j \text{ have same bday}).$

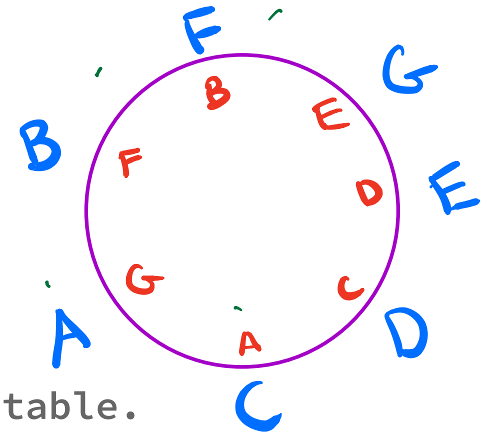
$= \left[\sum_{\ell=1}^{365} \boxed{\Pr(\text{student } i \text{ has bday day } \ell \cap \text{student } j \text{ has bday on day } \ell)}\right] = 365 \cdot \frac{1}{365} \cdot \frac{1}{365}$

$\boxed{\frac{1}{365} \cdot \frac{1}{365}}$

$= \frac{1}{365}$ ⇐



ROTATING THE TABLE



n people are sitting around a circular table.
There is a nametag in each place
Nobody is sitting in front of their own nametag.
Rotate the table by a random number k of positions between 1 and n-1 (equally likely).
X is the number of people that end up front of their own nametag.
What is E(X)?

$$\boxed{X = X_1 + X_2 + \dots + X_n}$$

$$X_i = \begin{cases} 1 & \text{if person } i \text{ ends up in front of their own nametag} \\ 0 & \text{o.w.} \end{cases}$$

$$E(X) = E(X_1) + E(X_2) + \dots + E(X_n) = \frac{n}{n-1}$$

$$E(X_i) = Pr(\text{person } i \text{ ends up in front of their name}) = \frac{1}{n-1}$$