Probability

3.1 Discrete Random Variables Basics

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Most slides by Alex Tsun

Agenda

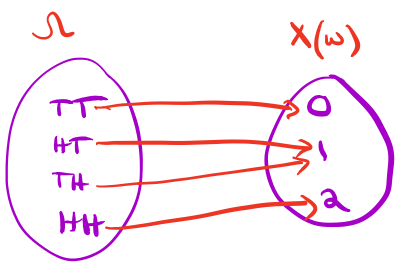
- Intro to Discrete Random Variables
- Probability Mass Functions
- Cumulative Distribution function
- Expectation

Flipping two coins

$$\Omega = \{HH, HT, TH, TT\}$$

Let X be the number of heads in two independent flips of a fair coin.

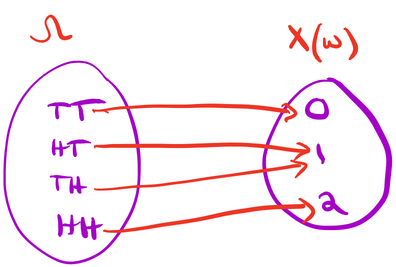
X is a function, $X: \Omega \to \mathbb{R}$ which takes outcomes $\omega \in \Omega$ and maps them to a number.



Random Variable

Suppose we conduct an experiment with sample space Ω. A **random variable (rv)** is a numeric function of the outcome, $X: \Omega \to \mathbb{R}$. That is, it maps outcomes $\omega \in \Omega$ to numbers, $\omega \mapsto X(\omega)$.

The set of possible values X can take on is its **range/support**, denoted Ω_X.



20 balls numbered 1..20

- Draw a subset of 3 uniformly at random.
- Let X = maximum of the numbers on the 3 balls.

$|\text{support of } X| = |\Omega_X|$

a) 20^3

b) 20

c) 18

d) $\binom{20}{3}$

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If Ω_X is finite or countably infinite (typically integers or a subset), X is a **discrete** random variable (drv). Else if Ω_X is uncountably large (the size of real numbers), X is **continuous** random variable.

Identify those RVs

For each of the following random variables, identify its range Ω_X and whether it is discrete or continuous.

RV Description	Range	drv or crv?
a) The number of heads in n flips of a fair coin.		
b) The number of people born this year.		
c) The number of flips of a fair coin up to and including my first head.		
d) The amount of time I wait for the next bus in seconds.		

Which cont.

a)
b)
c)
d)

Which has Range $\{1,2,3,\ldots\}$

a)
b)
c)
d)

Random Picture

Flipping two coins

$$\Omega = \{HH, HT, TH, TT\}$$

$$X(HH) = 2 \qquad X(HT) = 1 \qquad X(TH) = 1 \qquad X(TT) = 0$$

What is the support/range Ω_X? $\mathbf{\Omega_X = \{0, 1, 2\}}$

But what are the probabilities X takes on these values? For this, we define the **probability mass function (pmf)** of X, as $p_X\colon \Omega_X \to [0,1]$

$$p_X(k) = P(X = k)$$

Flipping two coins

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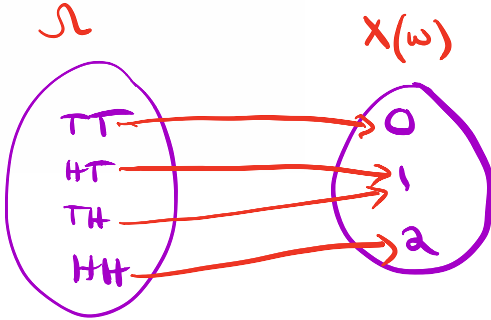
$$p_X(k) = \begin{cases} & k = 0 \\ & k = 1 \\ & k = 2 \end{cases}$$



Flipping two coins

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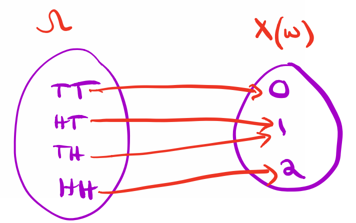
$$p_X(k) = P(X = k)$$

$$p_X(k) = \begin{cases} 1/4, & k = 0 \\ 1/2, & k = 1 \\ 1/4, & k = 2 \end{cases}$$



Probability Mass Function (PMF)

The **probability mass function (pmf)** of a discrete random variable X assigns probabilities to the possible values of the random variable. That is, $p_X: \Omega_X \to [0,1]$ where

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Note that $\{X = a\}$ for $a \in \Omega_X$ form a partition of Ω, since each outcome $\omega \in \Omega$ is mapped to exactly one number.

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Note that $\{X = a\}$ for $a \in \Omega_X$ form a partition of Ω, since each outcome $\omega \in \Omega$ is mapped to exactly one number. Hence,

$$\sum_{z \in \Omega_X} p_X(z) = 1$$

20 balls numbered 1..20

- Draw a subset of 3 uniformly at random.
- Let X = maximum of the numbers on the 3 balls.

- Pr (X = 20)

- Pr (X = 18)

a) $\binom{20}{2} / \binom{20}{3}$

b) $\binom{19}{2} / \binom{20}{3}$

c) $19^2 / \binom{20}{3}$

d) $19 \cdot 18 / \binom{20}{3}$

Cumulative distribution function(CDF)

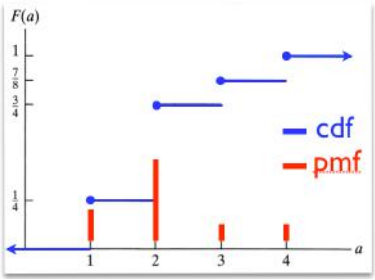
The **cumulative distribution function (CDF)** of a random variable $F_X(x)$ specifies for each possible real number x , the probability that $X \leq x$, that is

$$F_X(x) = P(X \leq x)$$

Ex: if X has probability mass function given by:

$$p(1) = \frac{1}{4} \quad p(2) = \frac{1}{2} \quad p(3) = \frac{1}{8} \quad p(4) = \frac{1}{8}$$

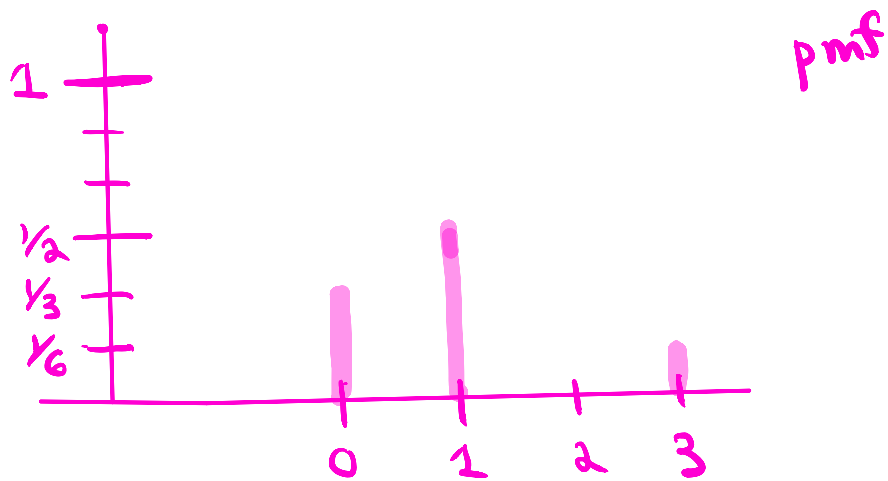
$$F(a) = \begin{cases} 0 & a < 1 \\ \frac{1}{4} & 1 \leq a < 2 \\ \frac{3}{4} & 2 \leq a < 3 \\ \frac{7}{8} & 3 \leq a < 4 \\ 1 & 4 \leq a \end{cases}$$



Homeworks of 3 students returned randomly

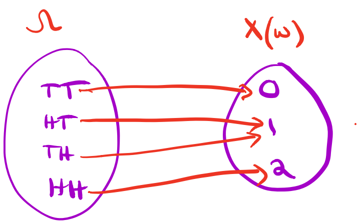
- Each permutation equally likely
- X: # people who get their own homework

Prob	Outcome w	X(w)
1/6	1 2 3	3
1/6	1 3 2	1
1/6	2 1 3	1
1/6	2 3 1	0
1/6	3 1 2	0
1/6	3 2 1	1



Flipping two coins

What is the *expected* number of heads in 2 flips of a fair coin?



Expectation

The **expectation/expected value/average** of a discrete random variable X is

$$E[X] = \sum_{\omega \in \Omega} X(\omega) P(\omega)$$

Or equivalently,

$$E[X] = \sum_{k \in \Omega_X} k \cdot p_X(k)$$

The interpretation is that we take an average of the values in Ω_X, but weighted by their probabilities.

Homeworks of 3 students returned randomly

- Each permutation equally likely
- X: # people who get their own homework
- What is E(X)?

Prob	Outcome w	X(w)
1/6	1 2 3	3
1/6	1 3 2	1
1/6	2 1 3	1
1/6	2 3 1	0
1/6	3 1 2	0
1/6	3 2 1	1

$$E(X) = \sum_{w \in \Omega} X(w) P(w)$$

$$= X(123)P(123) + X(132)P(132) + X(213)P(213) + X(321)P(321) + X(231)P(231) + X(312)P(312)$$

Flip a biased coin until get heads (flips independent)

With probability p of coming up heads
Keep flipping until the first Heads observed.

Let X be the number of flips until done.

- Pr(X = 1)
- Pr(X = 2)
- Pr(X = k)

a) p^k
b) $(1-p)^k$
c) $(1-p)^{k-1} p$
d) $p^{k-1}(1-p)$

Flip a biased coin until get heads (flips independent)

With probability p of coming up heads
Keep flipping until the first Heads observed.

Let X be the number of flips until done. What is E(X)?

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

$0 < x < 1$

Flip a biased coin independently

Probability p of coming up heads, n coin flips
X: number of Heads observed.

- Pr(X = k)

a) $\binom{n}{k} p^k$

b) $p^k (1-p)^{n-k}$

c) $\binom{n}{k} (1-p)^{n-k}$

d) $\binom{n}{k} p^k (1-p)^{n-k}$

Repeated coin flipping

Flip a biased coin with probability p of coming up Heads n times. Each flip independent of all others.

X is number of Heads.

What is E(X)?

$n=20 \quad p=\frac{1}{5}$

a) 1
b) 4
c) 5
d) 10

Repeated coin flipping

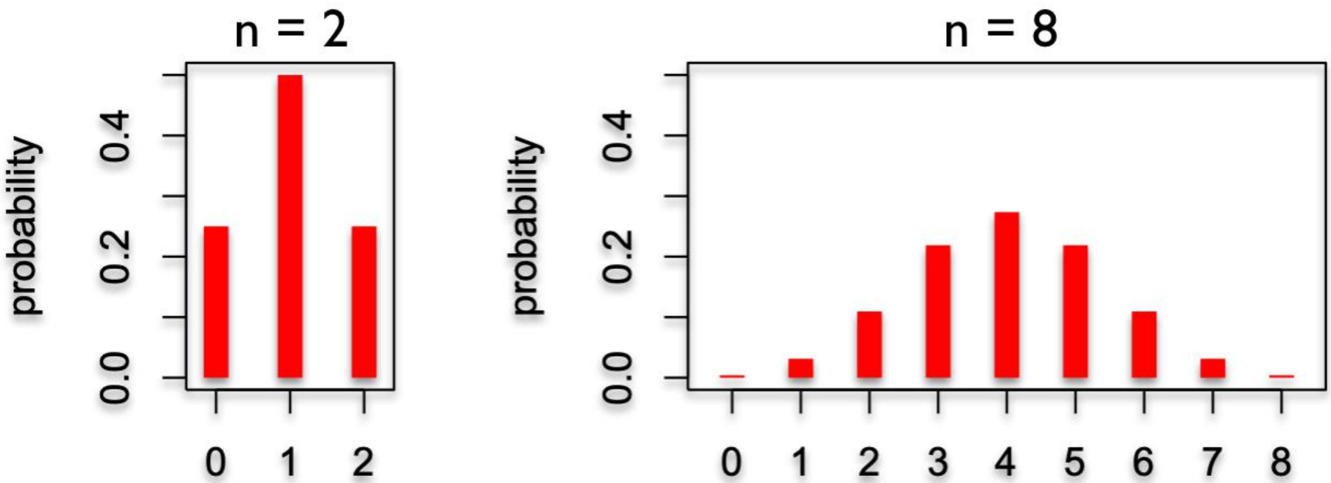
Flip a biased coin with probability p of coming up Heads n times.

X is number of Heads.

What is E(X)?

$$
\begin{aligned}
E[X] &= \sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i} \\
&= \sum_{i=1}^{n} i \binom{n}{i} p^i (1-p)^{n-i} \\
&= \sum_{i=1}^{n} n \binom{n-1}{i-1} p^i (1-p)^{n-i} \\
&= np \sum_{i=1}^{n} \binom{n-1}{i-1} p^{i-1} (1-p)^{n-i} \\
&= np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} \\
&= np(p + (1-p))^{n-1} = np
\end{aligned}
$$

Flip a Fair coin independently



Probability

3.2 More on Expectation

Alex Tsun

Agenda

- Linearity of Expectation (LoE)
- Law of the Unconscious Statistician (Lotus)

Linearity of Expectation (Idea)

Let's say you and your friend sell fish for a living.

- Every day you catch X fish, with $E[X] = 3$.
- Every day your friend catches Y fish, with $E[Y] = 7$.

How many fish do the two of you bring in ($Z = X + Y$) on an average day?

$E[Z] = E[X + Y] =$

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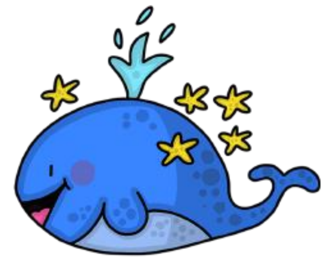

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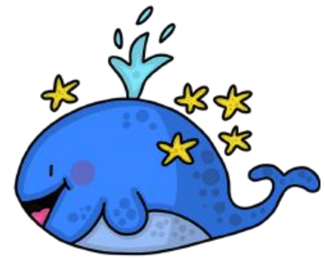
$$E[Z] = E[X + Y] = E[X] + E[Y] = 3 + 7 = 10$$

You can sell each fish for \$5 at a store, but you need to pay \$20 in rent. How much profit do you expect to make? $E[5Z - 20] =$

Linearity of Expectation (Idea)



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You can sell each fish for $5 at a store, but you need to pay $20 in rent. How much profit do you expect to make? $E[5Z - 20] = 5E[Z] - 20 = 5 \times 10 - 20 = 30$

Linearity of Expectation (LoE)

Linearity of Expectation: Let Ω be the sample space of an experiment, $X, Y: \Omega \to \mathbb{R}$ be (possibly "dependent") random variables both defined on Ω, and $a, b, c \in \mathbb{R}$ be scalars. Then,

$$E[X + Y] = E[X] + E[Y]$$

and

$$E[aX + b] = aE[X] + b$$

Combining them gives,

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

Linearity of Expectation (Proof)

$$E[X] + E[Y] = \sum_{\omega \in \Omega} X(\omega)P(\omega) + \sum_{\omega \in \Omega} Y(\omega)P(\omega)$$

$$= \sum_{\omega \in \Omega} \big(X(\omega) + Y(\omega)\big)P(\omega)$$

$$= \sum_{\omega \in \Omega} (X + Y)(\omega)\, P(\omega)$$

$$= E[X + Y]$$