Anonymous questions

Probability

3.1 Discrete Random Variables Basics

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Agenda

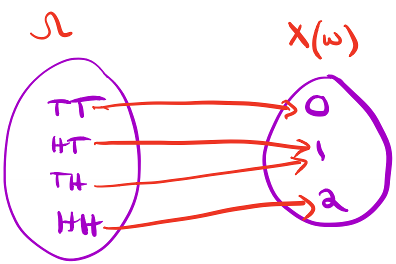
- Intro to Discrete Random Variables
- Probability Mass Functions
- Cumulative Distribution function
- Expectation

Flipping two coins

$$\Omega = \{HH, HT, TH, TT\}$$

Let X be the number of heads in two independent flips of a fair coin.

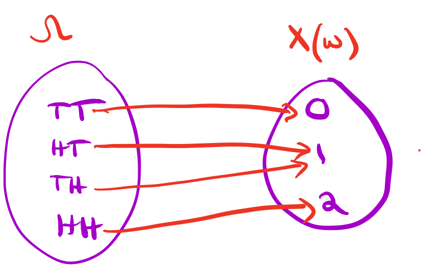
X is a function, $X: \Omega \to \mathbb{R}$ which takes outcomes $\omega \in \Omega$ and maps them to a number.



Random Variable

Suppose we conduct an experiment with sample space Ω. A **random variable (rv)** is a numeric function of the outcome, $X: \Omega \to \mathbb{R}$. That is, it maps outcomes $\omega \in \Omega$ to numbers, $\omega \mapsto X(\omega)$.

The set of possible values X can take on is its **range/support**, denoted Ω_X.



$\Omega_X = \{0, 1, 2\}$

20 BALLS NUMBERED 1..20

Ω: unordered subsets of 3 balls

$$P(\omega) = \frac{1}{|\Omega|} = \frac{1}{\binom{20}{3}}$$

- Draw a subset of 3 uniformly at random.
- Let X = maximum of the numbers on the 3 balls.

$$X(2,5,7) = 7$$

$$X(1,2,3) = 3$$

$$X(3,8,15) = 15$$

|support of X| = $|\Omega_X|$

$$\Omega_X = \{3,4,5,\ldots,20\}$$

a) 20^3

b) 20

c) $\boxed{18}$

d) $\binom{20}{3}$

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If Ω_X is finite or countably infinite (typically integers or a subset), X is a **discrete** random variable (drv). Else if Ω_X is uncountably large (the size of real numbers), X is **continuous** random variable.

Identify those RVs

For each of the following random variables, identify its range Ω_X and whether it is discrete or continuous.

	RV Description	Range	drv or crv?
a)	The number of heads in n flips of a fair coin.	$\{0,1,2,\dots,n\}$	drv
b)	The number of people born this year.	$\{0,1,2,\dots$	drv
c)	The number of flips of a fair coin up to and including my first head.	$\{1,2,3,\dots\}$	drv
d)	The amount of time I wait for the next bus in seconds.	$[0,\infty)$	crv

Which cont.

a)
b)
c)
d)

Which has Range $\{1,2,3,\dots\}$

a)
b)
$\boxed{c)}$
d)

Random Picture

Flipping two coins

$$\Omega = \{HH, HT, TH, TT\}$$

$$X(HH) = 2 \qquad X(HT) = 1 \qquad X(TH) = 1 \qquad X(TT) = 0$$

What is the support/range Ω_X? $\boldsymbol{\Omega_X = \{0, 1, 2\}}$

But what are the probabilities X takes on these values? For this, we define the **probability mass function (pmf)** of X, as $p_X : \Omega_X \to [0,1]$

$$\boxed{p_X(k)} = P(X = k)$$

$$\{X = k\} = \{\omega \mid X(\omega) = k\}$$

Flipping two coins

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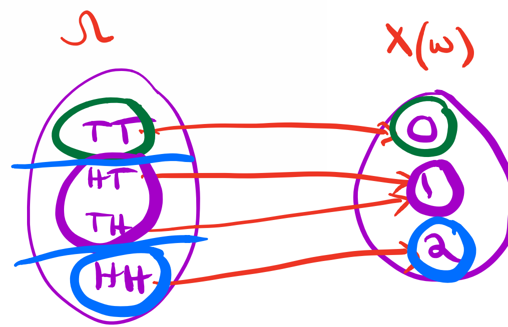
$$p_X(k) = P(X = k)$$

$$\boxed{p_X(k) = \begin{cases} 1/4 & k = 0 \\ 1/2 & k = 1 \\ 1/4 & k = 2 \end{cases}}$$



Flipping two coins

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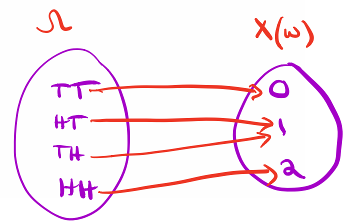

Probability Mass Function (PMF)

The **probability mass function (pmf)** of a discrete random variable X assigns probabilities to the possible values of the random variable. That is, $p_X \colon \Omega_X \to [0,1]$ where

$$p_X(k) = P(X = k) = P(\{\omega \mid X(\omega) = k\})$$

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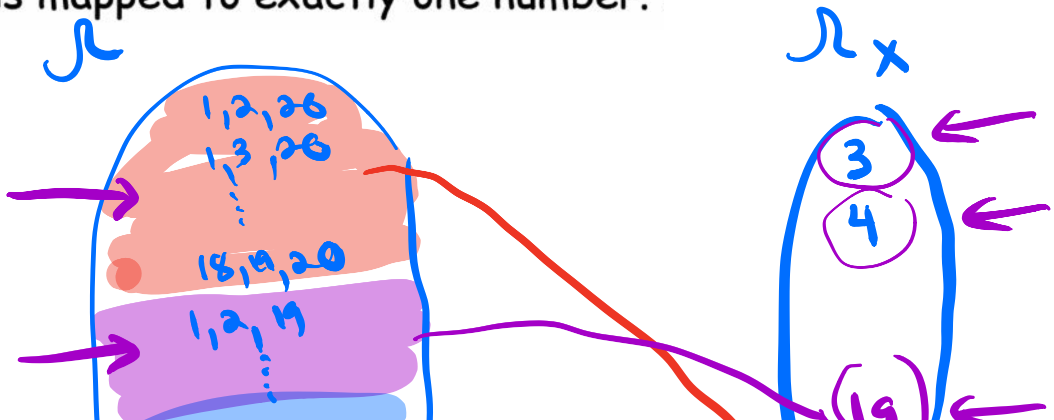

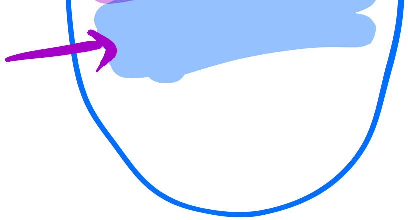

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$$p_X(k) = P(X = k)$$

Note that $\{X = a\}$ for $a \in \Omega_X$ form a partition of Ω, since each outcome $\omega \in \Omega$ is mapped to exactly one number. Hence,

$$\sum_{z \in \Omega_X} p_X(z) = 1$$

$P(X = z) = P(\{w | X(w) = z\})$

20 balls numbered 1..20

- Draw a subset of 3 uniformly at random.
- Let X = maximum of the numbers on the 3 balls.

- Pr (X = 20)

- Pr (X = 18)

a) $\binom{20}{2} / \binom{20}{3}$

$\boxed{\text{b) } \binom{19}{2} / \binom{20}{3}}$

c) $19^2 / \binom{20}{3}$

d) $19 \cdot 18 / \binom{20}{3}$

$\{X=20\} = \{w \mid \text{max ball number on a ball is } 20\}$

$$P(X=20) = \frac{|\{X=20\}|}{\binom{20}{3}} = \frac{\binom{19}{2}}{\binom{20}{3}}$$

$\binom{k-1}{2} / \binom{20}{3}$ $k=3$ 20

$P_X(k) = P(X=k) = \begin{cases} \cdots \\ 0 & \text{otherwise.} \end{cases}$

Cumulative distribution function(CDF)

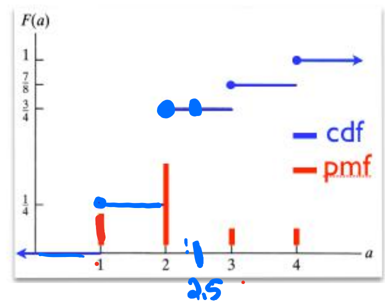
The **cumulative distribution function (CDF)** of a random variable $F_X(x)$ specifies for each possible real number x, the probability that $X \leq x$, that is

$$F_X(x) = P(X \leq x)$$

Ex: if X has probability mass function given by:

$p(1) = \frac{1}{4} \quad p(2) = \frac{1}{2} \quad p(3) = \frac{1}{8} \quad p(4) = \frac{1}{8}$

$$F(a) = \begin{cases} 0 & a < 1 \\ \frac{1}{4} & 1 \leq a < 2 \\ \frac{3}{4} & 2 \leq a < 3 \\ \frac{7}{8} & 3 \leq a < 4 \\ 1 & 4 \leq a \end{cases}$$



Homeworks of 3 students returned randomly

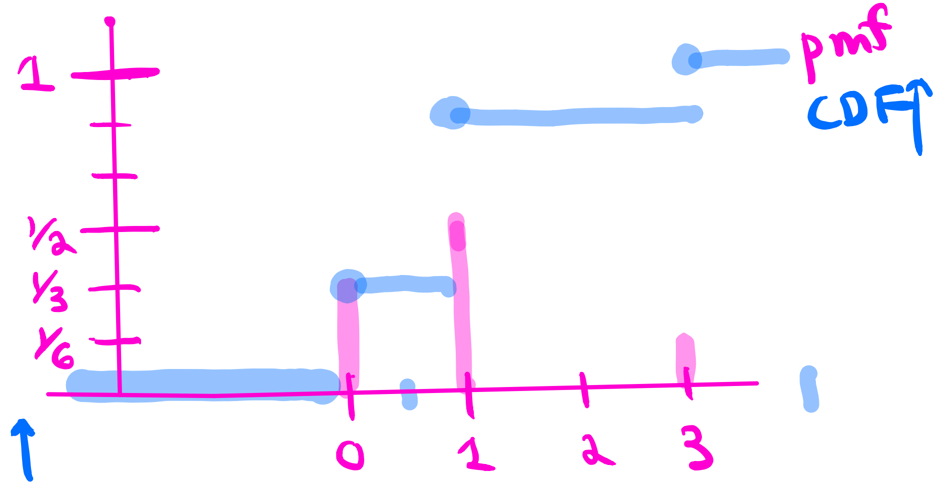
- Each permutation equally likely
- X: # people who get their own homework

$$P_X(x) = \begin{cases} \frac{1}{3} & x=0 \\ \frac{1}{2} & x=1 \\ \frac{1}{6} & x=3 \end{cases}$$

Prob	Outcome w	X(w)
1/6	1 2 3	3
1/6	1 3 2	1
1/6	2 1 3	1
1/6	2 3 1	0
1/6	3 1 2	0
1/6	3 2 1	1

$\forall x$

$\Pr(X \le x)$



Flipping two coins

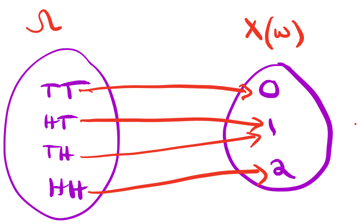
What is the *expected* number of heads in 2 flips of a fair coin? 1

$$E(X) = 0 \cdot Pr(X=0) + 1 \cdot Pr(X=1) + 2 \cdot P(X=2)$$

expectation or expected value

$$= 0 \cdot \tfrac{1}{4} + 1 \cdot \tfrac{1}{2} + 2 \cdot \tfrac{1}{4} = 1$$

$\Downarrow$



$$E(X) = X(TT)P(TT) + X(HT)P(HT) + X(TH)P(TH) + X(HH)P(HH)$$

$$= 0 \cdot \tfrac{1}{4} + 1 \cdot \tfrac{1}{4} + 1 \cdot \tfrac{1}{4} + 2 \cdot \tfrac{1}{4}$$

Expectation

The **expectation/expected value/~~average~~** of a discrete random variable X is

$$E[X] = \sum_{\omega \in \Omega} X(\omega) P(\omega)$$

Or equivalently,

$$\boxed{E[X] = \sum_{k \in \Omega_X} k \cdot p_X(k)}$$

The interpretation is that we take an average of the values in Ω_X, but weighted by their probabilities.

Homeworks of 3 students returned randomly

- Each permutation equally likely
- X: # people who get their own homework
- What is E(X)?

Prob	Outcome w	X(w)
1/6	1 2 3	3
1/6	1 3 2	1
1/6	2 1 3	1
1/6	2 3 1	0
1/6	3 1 2	0
1/6	3 2 1	1

$$\boxed{\Omega_X = \{0, 1, 3\}}$$

$$E(X) = 0 \cdot \boxed{\overbrace{P(X=0)}^{P_X(0)}} + 1 \cdot \overbrace{P(X=1)}^{P_X(1)} + 3\,\overbrace{P(X=3)}^{P_X(3)}$$

$$= 0\,\tfrac{1}{3} + 1 \cdot \tfrac{1}{2} + 3 \cdot \tfrac{1}{6}$$

$$= 1$$

$$E(X) = \sum_{\omega \in \Omega} X(\omega) P(\omega)$$

$$= X(123)P(123) + X(132)P(132) + X(213)P(213) + X(321)P(321) + X(231)P(231) + X(312)P(312)$$

$$3 \cdot \tfrac{1}{6} \qquad 1 \cdot \tfrac{1}{6} \qquad + \ldots$$

$n = 1000$ $\quad E(X) = \sum_{k=0}^{n} k \, \boxed{\underline{Pr(X=k)}}$

Flip a biased coin until get heads (flips independent)

$\Rightarrow \Omega = \{ \underset{1}{H}, \underset{2}{TH}, \underset{3}{TTH}, TTTH, \ldots \}$ values of X

With probability p of coming up heads
Keep flipping until the first Heads observed.

Let X be the number of flips until done.

- Pr(X = 1) $= p$
- Pr(X = 2) $= (1-p)p$
- Pr(X = k) $= (1-p)^{k-1} p$

a) p^k
b) $(1-p)^k$
$\Rightarrow$ c) $(1-p)^{k-1} p$
d) $p^{k-1}(1-p)$

$(\Omega, P(\cdot))$ $\quad X: \Omega \to \mathbb{R}$

$X(\omega)$ is # flips till Heads outcome

Ω_X: set of values X takes
range/support

Flip a biased coin until get heads (flips independent)

$p = \frac{1}{20}$

With probability p of coming up heads
Keep flipping until the first Heads observed.

Let X be the number of flips until done. What is E(X)?

$= \frac{1}{p}$.

$$E(X) = \sum_{k=1}^{\infty} k \Pr(X=k)$$

$$= \sum_{k=1}^{\infty} k (1-p)^{k-1} p = \frac{1}{p}$$

extra:

$$\boxed{\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad 0<x<1} \overset{\frac{d}{dx}}{\Longrightarrow} \sum_{k=1}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2}$$