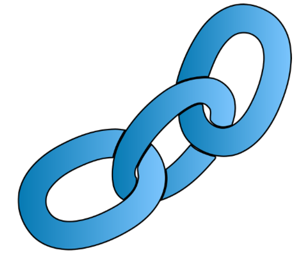
# Probability
## 2.3 Independence

Anna Karlin
Most slides by Alex Tsun

# Agenda

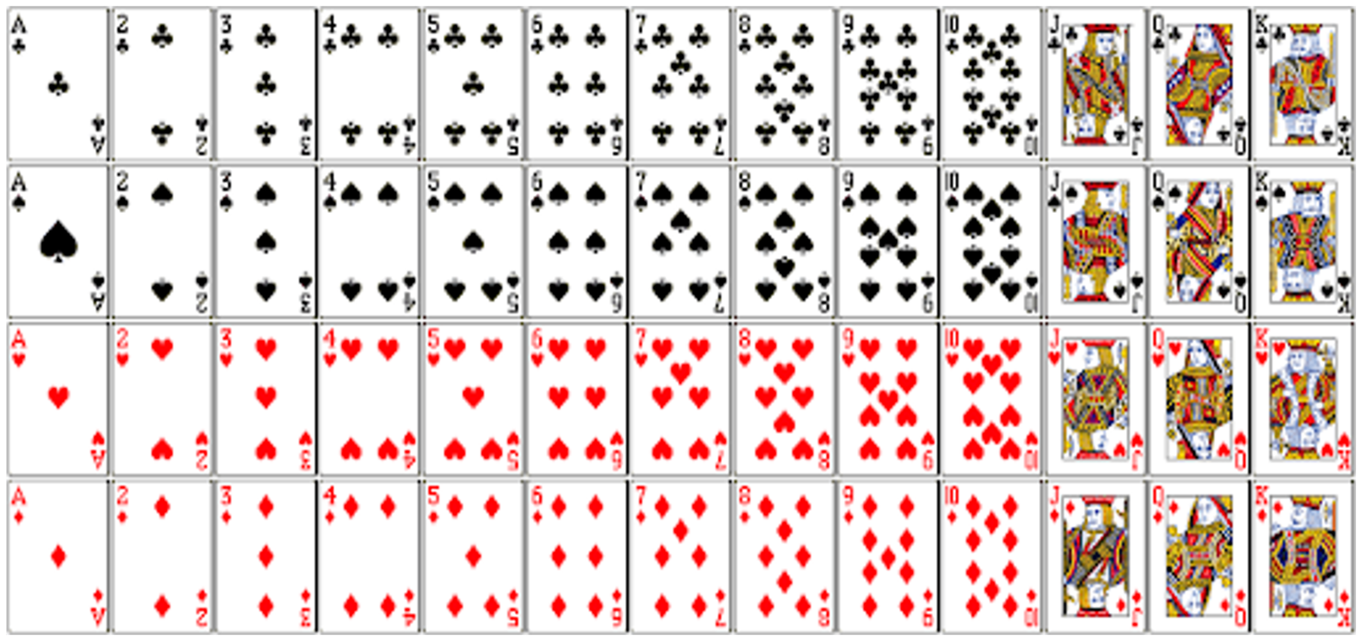
- Chain Rule
- Independence
- Conditional Independence

# Chain Rule (Idea)



Have a Standard 52-Card Deck.

- 4 Suits (Clubs, Diamonds, Hearts, Spades)
- 13 ranks (A, 2, 3, …, 9, 10, J, Q, K)



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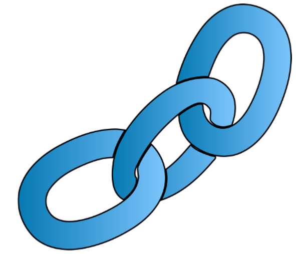
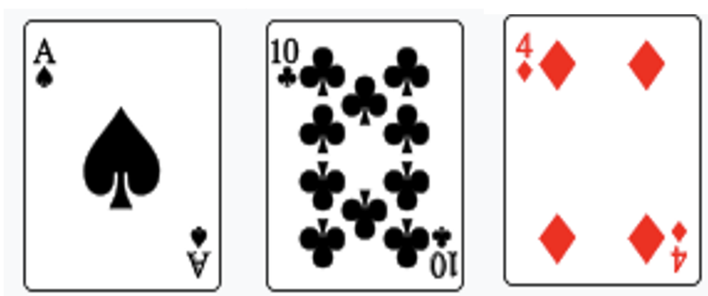


Have a Standard 52-Card Deck. Shuffle It, and draw the top 3 cards.

What is P ( 

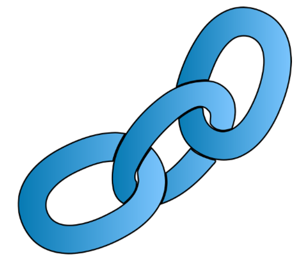
) = P(**A**, **B**, **C**)?

**A**: Ace of Spades First

**B**: 10 of Clubs Second

**C**: 4 of Diamonds Third

# Chain Rule (Idea)



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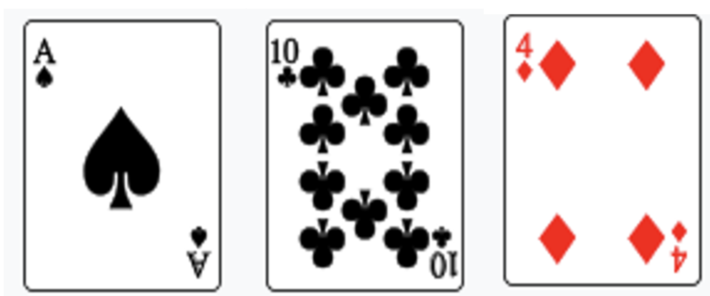
What is P (  ) = P(A, B, C)?

$$\frac{1}{52} \cdot \frac{1}{51} \cdot \frac{1}{50}$$

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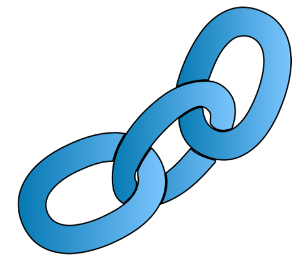

Have a Standard 52-Card Deck. Shuffle It, and draw the top 3 cards. (uniform probability space).

A: Ace of Spades First

B: 10 of Clubs Second

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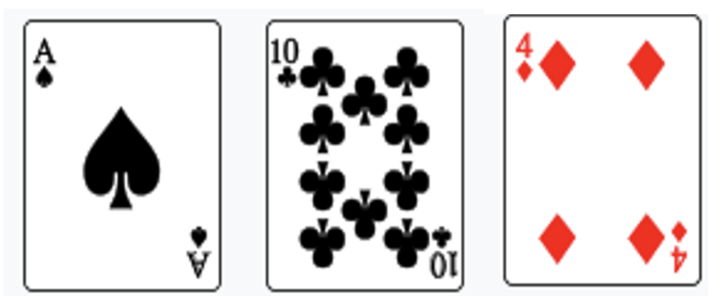
What is P (  ) = P(A, B, C)?

$$\frac{1}{52} \cdot \frac{1}{51} \cdot \frac{1}{50}$$

$$P(A) \cdot P(B|A) \cdot P(C|A,B)$$

# Chain Rule

**Chain Rule:** Let $A_1, \dots, A_n$ be events with nonzero probability. Then,

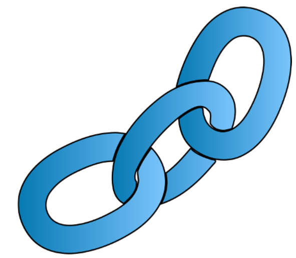
$$P(A_1, \dots, A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1, A_2) \dots P(A_n|A_1, \dots, A_{n-1})$$

In the case of two events $A, B$,

$$P(A, B) = P(A)P(B|A)$$

An easy way to remember this formula: we need to do $n$ tasks, so we can perform them one at a time, conditioning on what we've done so far.

# Chain Rule (Idea)



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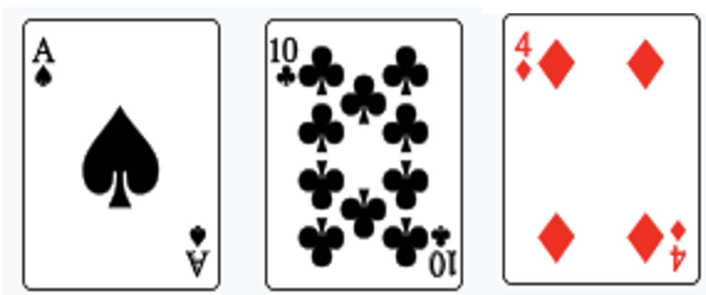

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$$P(A) \cdot P(B|A) \cdot P(C|A,B)$$

# Fun with cards

Two people, A and B, are playing the following game.
A 6-sided die is thrown and each time it's thrown, regardless of the history, it is equally likely to show any of the six numbers
If it shows 5, A wins.
If it shows 1, 2 or 6, B wins.
Otherwise, they play a second round and so on.

What is Pr(A wins on $4^{th}$ round)?

# The need for independence

**Quick question**: In general, is

$$P(A, B) = P(A)P(B)?$$

# The need for independence

**Quick question**: In general, is

$$P(A, B) = P(A)P(B)?$$

The chain rule says

$$P(A, B) = P(A)P(B|A)$$

So no, unless the special case when $P(B|A) = P(B)$. This case is so important it has a name.

# Independence

**Independence:** Events $A, B$ are independent if any of the three equivalent conditions hold:

1. $P(A|B) = P(A)$
2. $P(B|A) = P(B)$
3. $P(A, B) = P(A)P(B)$

# Independence

- Toss a coin 3 times. Each of 8 outcomes equally likely.

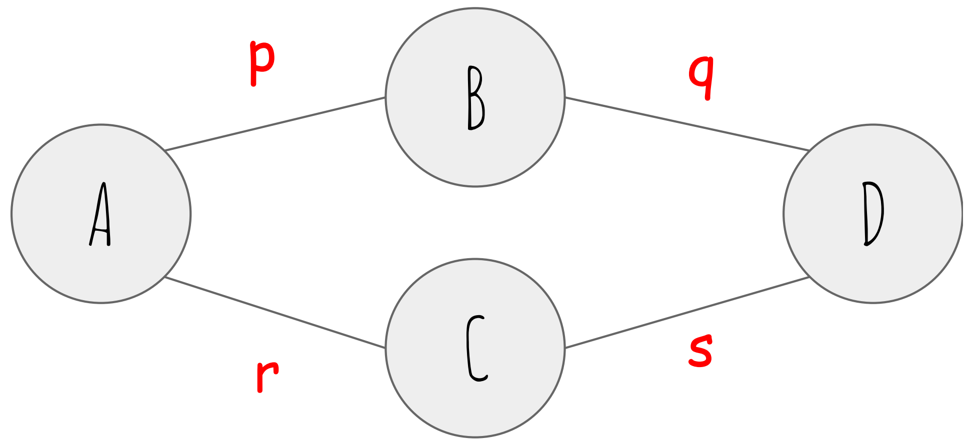
Define

- A = {at most one T} = {HHH, HHT, HTH, THH}
- B = {at most 2 Heads}= $\{HHH\}^c$

- Are A and B independent?

# Network Communication

Each link works with the probability given, **independently**. What's the probability A and D can communicate?



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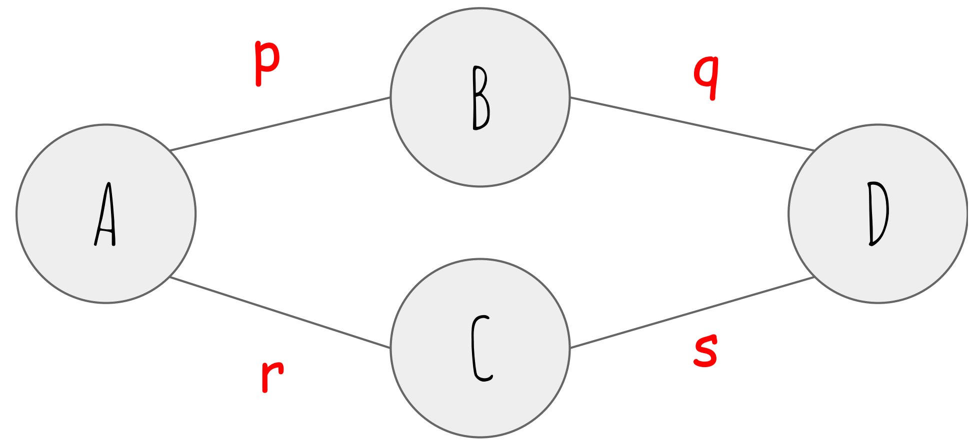


$$P(top) = P(AB \cap BD) = P(AB)P(BD) = pq$$
$$P(bottom) = P(AC \cap CD) = P(AC)P(CD) = rs$$

$$\begin{aligned} P(top \cup bottom) &= P(top) + P(bottom) - P(top \cap bottom) \\ &= P(top) + P(bottom) - P(top)P(bottom) \\ &= pq + rs - pqrs \end{aligned}$$

# Using independence to define a probabilistic model

- We can **define** our probability model via independence.

- Example: suppose a biased coin comes up heads with probability 2/3, independent of other flips.

- Sample space: sequences of 3 coin tosses.

- Pr (HHH)=?
- Pr (TTT) = ?
- Pr (HHT) = ?
- Pr (HTH) = ?
- Pr (2 heads) = ?

# Probability
## 3.1 Discrete Random Variables Basics

Anna Karlin
Most slides by Alex Tsun

# Agenda

- Intro to Discrete Random Variables
- Probability Mass Functions
- Cumulative Distribution function

# Flipping two coins


$$\Omega = \{HH, HT, TH, TT\}$$

Let $X$ be the number of heads in two independent flips of a fair coin.

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Let $X$ be the number of heads in two independent flips of a fair coin.

$X$ is a function, $X: \Omega \to \mathbb{R}$ which takes outcomes $\omega \in \Omega$ and maps them to a number.

# RANDOM VARIABLE

Suppose we conduct an experiment with sample space $\Omega$. A **random variable (rv)** is a numeric function of the outcome, $X: \Omega \to \mathbb{R}$. That is, it maps outcomes $\omega \in \Omega$ to numbers, $\omega \mapsto X(\omega)$.

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The set of possible values $X$ can take on is its **range/support**, denoted $\Omega_X$.

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- Let X = maximum of the numbers on the 3 balls.

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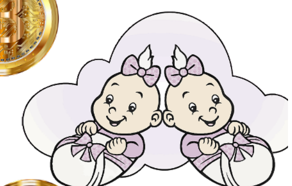
If $\Omega_X$ is finite or countably infinite (typically integers or a subset), $X$ is a **discrete** random variable (drv). Else if $\Omega_X$ is uncountably large (the size of real numbers), $X$ is **continuous** random variable.

# Identify those RVs

For each of the following random variables, identify its range $\Omega_X$ and whether it is discrete or continuous.

| RV Description | Range | drv or crv? |
|---|---|---|
| The number of heads in $n$ flips of a fair coin. | | |
| The number of people born this year. | | |
| The number of flips of a fair coin up to and including my first head. | | |
| The amount of time I wait for the next bus in seconds. | | |



# Random Picture

# Flipping two coins

$$\Omega = \{HH, HT, TH, TT\}$$

$$X(HH) = 2 \qquad X(HT) = 1 \qquad X(TH) = 1 \qquad X(TT) = 0$$

What is the support/range $\Omega_X$? $\boldsymbol{\Omega_X = \{0, 1, 2\}}$

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But what are the probabilities $X$ takes on these values? For this, we define the **probability mass function (pmf)** of $X$, as $p_X: \Omega_X \to [0,1]$

$$p_X(k) = P(X = k)$$

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$$p_X(k) = P(X = k)$$

$$p_X(k) = \begin{cases} & k = 0 \\ & k = 1 \\ & k = 2 \end{cases}$$

# Flipping two coins

 

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$$p_X(k) = P(X = k)$$

$$p_X(k) = \begin{cases} 1/4, & k = 0 \\ 1/2, & k = 1 \\ 1/4, & k = 2 \end{cases}$$

# Probability Mass Function (PMF)

The **probability mass function (pmf)** of a discrete random variable $X$ assigns probabilities to the possible values of the random variable. That is, $p_X: \Omega_X \to [0,1]$ where

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Note that $\{X = a\}$ for $a \in \Omega_X$ form a partition of $\Omega$, since each outcome $\omega \in \Omega$ is mapped to exactly one number. Hence,

$$\sum_{z \in \Omega_X} p_X(z) = 1$$

# 20 balls numbered 1..20

- Draw a subset of 3 uniformly at random.
- Let X = maximum of the numbers on the 3 balls.

- Pr (X =20)

- Pr (X = 18)

- Pr (X < 17)

# Flip a biased coin until get heads (flips independent)

With probability p of coming up heads
Keep flipping until the first Heads observed.

Let X be the number of flips until done.

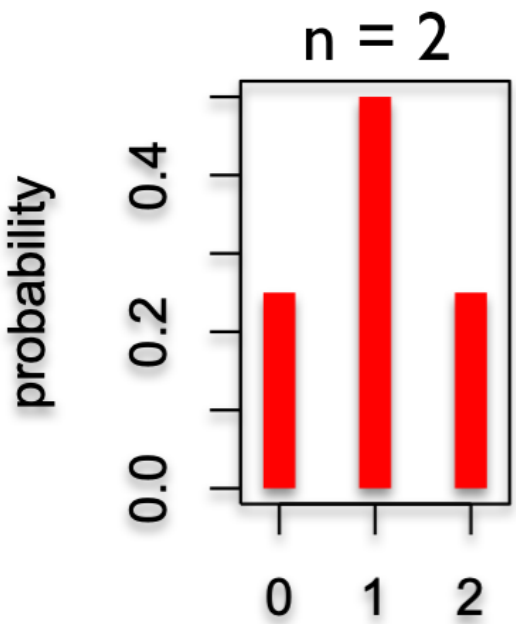
- Pr(X = 1)
- Pr(X = 2)
- Pr(X = k)

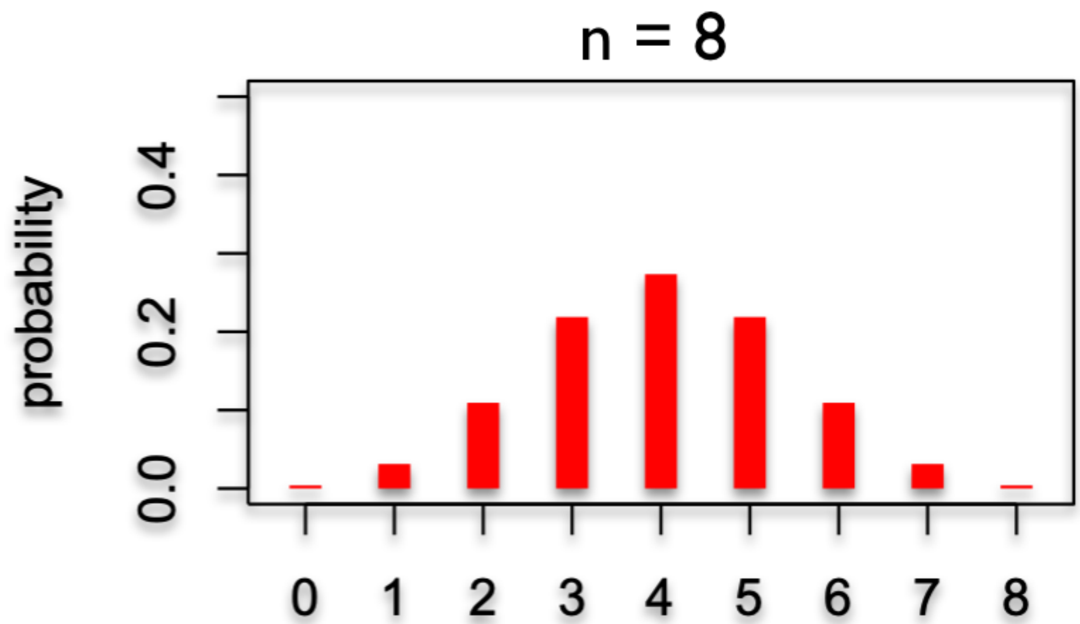
# Flip a biased coin independently

Probability p of coming up heads, n coin flips
X: number of Heads observed.

- Pr(X = k)

# Flip a biased coin independently



# Cumulative distribution function(CDF)

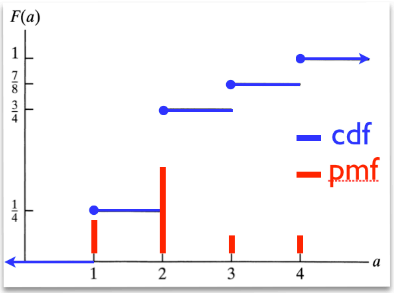
The **cumulative distribution function (CDF)** of a random variable $F_X(x)$ specifies for each possible real number $x$, the probability that $X \leq k$, that is

$$F_X(x) = P(X \leq k)$$

Ex: if $X$ has probability mass function given by:

$$p(1) = \tfrac{1}{4} \qquad p(2) = \tfrac{1}{2} \qquad p(3) = \tfrac{1}{8} \qquad p(4) = \tfrac{1}{8}$$

$$F(a) = \begin{cases} 0 & a < 1 \\ \frac{1}{4} & 1 \leq a < 2 \\ \frac{3}{4} & 2 \leq a < 3 \\ \frac{7}{8} & 3 \leq a < 4 \\ 1 & 4 \leq a \end{cases}$$



# Homeworks of 3 students returned randomly

- Each permutation equally likely
- X: # people who get their own homework

| Prob | Outcome w | X(w) |
|---|---|---|
| 1/6 | 1 2 3 | 3 |
| 1/6 | 1 3 2 | 1 |
| 1/6 | 2 1 3 | 1 |
| 1/6 | 2 3 1 | 0 |
| 1/6 | 3 1 2 | 0 |
| 1/6 | 3 2 1 | 1 |