Probability

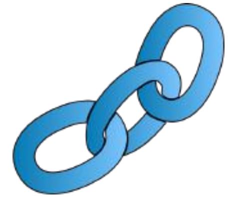
2.3 Independence

Anna Karlin
Most slides by Alex Tsun

Agenda

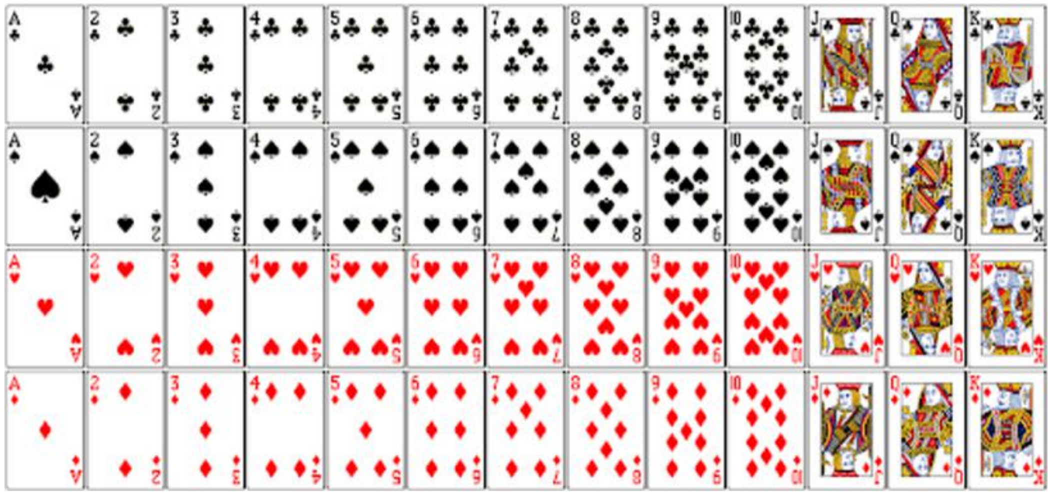
- Chain Rule
- Independence
- Conditional Independence not in class.

Chain Rule (Idea)

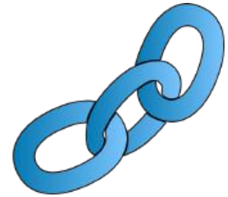


Have a Standard 52-Card Deck.

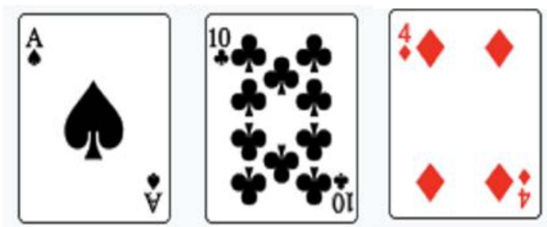
- 4 Suits (Clubs, Diamonds, Hearts, Spades)
- 13 ranks (A, 2, 3, …, 9, 10, J, Q, K)



Chain Rule (Idea)



Have a Standard 52-Card Deck. Shuffle It, and draw the top 3 cards.

What is P (

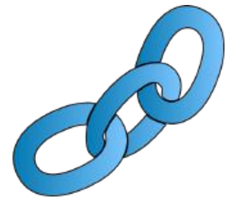
) = P(**A**, **B**, **C**)?

A: Ace of Spades First

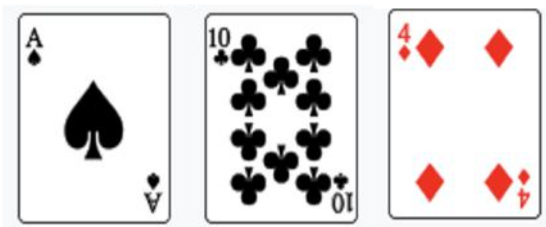
B: 10 of Clubs Second

C: 4 of Diamonds Third

Chain Rule (Idea)



Have a Standard 52-Card Deck. Shuffle It, and draw the top 3 cards.

What is P () = P(A, B, C)?

A: Ace of Spades First

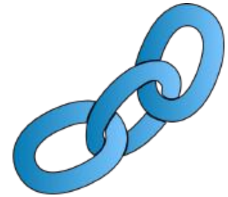
B: 10 of Clubs Second

C: 4 of Diamonds Third

$$\frac{1}{52} \cdot \frac{1}{51} \cdot \frac{1}{50}$$

$P(A) \cdot P(B|A) \cdot P(C|A \cap B)$

Chain Rule (Idea)

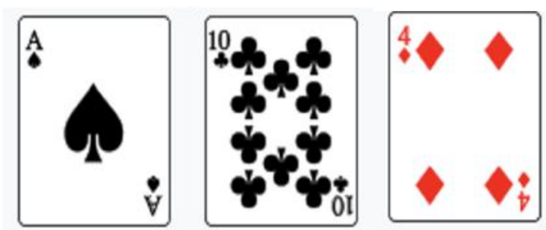


Have a standard 52-card deck. Shuffle it, and draw the top 3 cards. (uniform probability space).

A: Ace of Spades First

B: 10 of Clubs Second

C: 4 of Diamonds Third

What is P() = P(A, B, C)?

$$\frac{1}{52} \cdot \frac{1}{51} \cdot \frac{1}{50}$$

$$P(A) \cdot P(B|A) \cdot P(C|\underbrace{A,B}_{A\cap B})$$

$$A,B,C \triangleq A\cap B\cap C$$

Chain Rule

Chain Rule: Let $A_1, \ldots, A_n$ be events with nonzero probability. Then,

$$P(A_1, \ldots, A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1, A_2) \ldots P(A_n|A_1, \ldots, A_{n-1})$$

In the case of two events A, B,

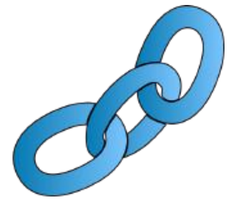
$$P(A, B) = P(A)P(B|A)$$

$P(B|A) = \frac{P(A \cap B)}{P(A)}$

An easy way to remember this formula: we need to do n tasks, so we can perform them one at a time, conditioning on what we've done so far.

Chain Rule (Idea)

$Pr(E) = \frac{|E|}{|\Omega|}$



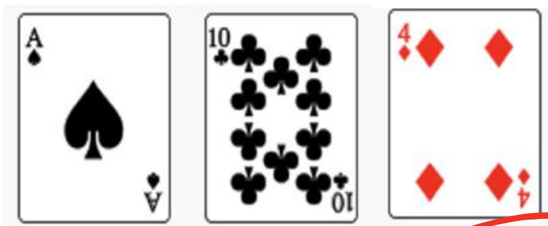
Have a standard 52-card deck. Shuffle it, and draw the top 3 cards. (Uniform probability space).

$|\Omega| = 52 \cdot 51 \cdot 50$

A: Ace of Spades First

B: 10 of Clubs Second

C: 4 of Diamonds Third

What is P () = P(A, B, C)?

$$\frac{1}{52} \cdot \boxed{\frac{1}{51}} \cdot \boxed{\frac{1}{50}}$$

$$P(A) \cdot \boxed{P(B|A)} \cdot \boxed{P(C|A,B)}$$

$P(C|A\cap B)$
$= P(C|B\cap A)$

$P(B) = P(2^{nd}$ card is 10 clubs$)$

a) $\frac{1}{51}$

b) $\frac{1}{52}$

c) $\frac{1}{13}$

d) $\frac{1}{4}$

$\frac{}{51} \cdot \frac{10C}{50}$

$|B|$

$P(B) \neq P(B|A)$

$$P(B) = \frac{\cancel{51} \cdot \cancel{50}}{52 \cdot \cancel{51} \cdot \cancel{50}} = \frac{1}{52}$$

$$P(B|A) = \frac{Pr(A \cap B)}{P(A)} = \frac{|A \cap B|}{|A|} = \frac{50}{51 \cdot 50} = \frac{1}{51}$$

Fun with Cards

Two people, A and B, are playing the following game.
A 6-sided die is thrown and each time it's thrown, regardless of the history, it is equally likely to show any of the six numbers
If it shows 5, A wins.
If it shows 1, 2 or 6, B wins.
Otherwise, they play a second round and so on.

What is Pr(A wins on 4th round)?

will be on section worksheet.

use chain rule.

The need for independence

Quick question: In general, is

$$P(A \cap B) = P(A,B) = P(A)P(B)?$$

The need for independence

Quick question: In general, is

$$P(A, B) = P(A)P(B)?$$

The chain rule says

$$P(A, B) = P(A)P(B|A)$$

So no, unless the special case when $P(B|A) = P(B)$. This case is so important it has a name.

Independence

Independence: Events A, B are independent if any of the three equivalent conditions hold:

1. $P(A|B) = P(A)$
2. $P(B|A) = P(B)$
3. $P(A, B) = P(A)P(B)$

Independence

- Toss a coin 3 times. Each of 8 outcomes equally likely.

Define

- A = {at most one T} = {HHH, HHT, HTH, THH}
- B = {at most 2 Heads}= {HHH}^{c}

- Are A and B independent?

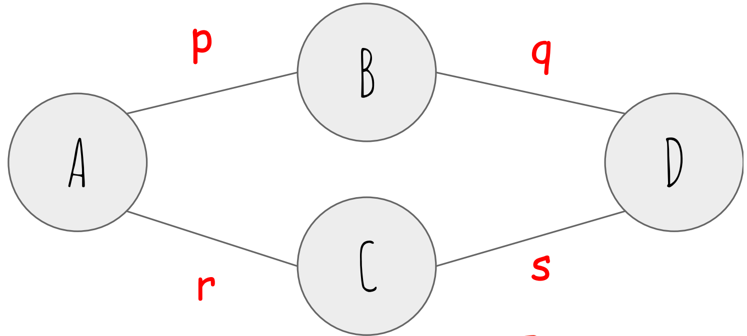
$A \cap B = \{HHT, HTH, THH\}$

$P(A)\,P(B) \neq P(A \cap B)$

not indep.

	Pr(A)	Pr(B)	Pr(A∩B)
a)	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{16}$
b)	$\frac{1}{2}$	$\frac{7}{8}$	$\frac{7}{16}$
c)	$\frac{1}{2}$	$\frac{7}{8}$	$\frac{3}{8}$
d)	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{21}{32}$

Network Communication



Each link works with the probability given, independently. What's the probability A and D can communicate?

T: top path works

B: bottom path works.

$P(T) = P(AB \cap BD)$

$= P(AB)\,P(BD)$

$= pq \Leftarrow$

$P(B) = r \cdot s$

$P(\exists \text{ working path}) = P(T \cup B) = P(T) + P(B) - P(T \cap B)$

$= pq + rs - \underbrace{P(T)\,P(B)}_{pqrs}$

$1 - \Pr(\text{no path works})$

$\Pr(\bar{T} \cap \bar{B}) = P(\bar{T})P(\bar{B}) = (1-pq)(1-rs)$

$$\boxed{P(C \cup D) = P(C) + P(D) - P(C \cap D)}$$

when C & D are independent

$P(C \cap D) = P(C)P(D)$

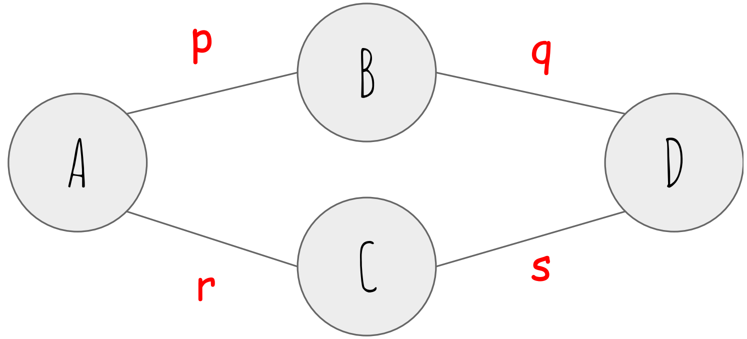
C, D are mutually exclusive but

a) indep

b) not indep

$P(D) \neq P(D|C) = 0$

Network Communication



Each link works with the probability given, **independently**. What's the probability A and D can communicate?

$$P(top) = P(AB \cap BD) = P(AB)P(BD) = pq$$
$$P(bottom) = P(AC \cap CD) = P(AC)P(CD) = rs$$

$$\begin{aligned} P(top \cup bottom) &= P(top) + P(bottom) - P(top \cap bottom) \\ &= P(top) + P(bottom) - P(top)P(bottom) \\ &= pq + rs - pqrs \end{aligned}$$

Using independence to define a probabilistic model

- We can **define** our probability model via independence.
- Example: suppose a biased coin comes up heads with probability 2/3, independent of other flips.
- Sample space: sequences of 3 coin tosses.

$\Omega = \{H, T\}^3$

$= P(H)P(H)P(H)$

- Pr (HHH)=? $\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \left(\frac{2}{3}\right)^3$
- Pr (TTT) = ? $\left(\frac{1}{3}\right)^3$
- Pr (HHT) = ? $\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} = \left(\frac{2}{3}\right)^2 \frac{1}{3}$
- Pr (HTH) = ? $\frac{2}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} = \left(\frac{2}{3}\right)^2 \frac{1}{3}$
- Pr (2 heads) = ?

Pr(2 heads)

a) $\left(\frac{2}{3}\right)^2 \frac{1}{3}$

b) $\frac{2}{3}$

c) $\boxed{\left(\frac{2}{3}\right)^2}$

d) $\left(\frac{1}{3}\right)^2$

$Pr(\{HHT, HTH, THH\})$

$= P(HHT) + P(HTH) + P(THH)$

$= 3 \left(\frac{2}{3}\right)^2 \frac{1}{3} = \left(\frac{2}{3}\right)^2$

$\binom{3}{2}\left(\frac{2}{3}\right)^2 \frac{1}{3}$

$\shortparallel$

$3 \left(\frac{2}{3}\right)^2 \cdot \frac{1}{3}$

$\boxed{Pr(H_1 T_2 H_3)}$

$= \underline{P(H_1)}\,\underline{P(T_2)}\,\underline{P(H_3)}$

$= \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{2}{3}$

$= \left(\frac{2}{3}\right)^2 \frac{1}{3}$

$$\boxed{P(E) = \sum_{\omega \in E} Pr(\omega)}$$

Probability

3.1 Discrete Random Variables Basics

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Most slides by Alex Tsun

Agenda

- Intro to Discrete Random Variables
- Probability Mass Functions
- Cumulative Distribution function

Flipping two coins

$$\Omega = \{HH, HT, TH, TT\}$$

Let X be the number of heads in two independent flips of a fair coin.

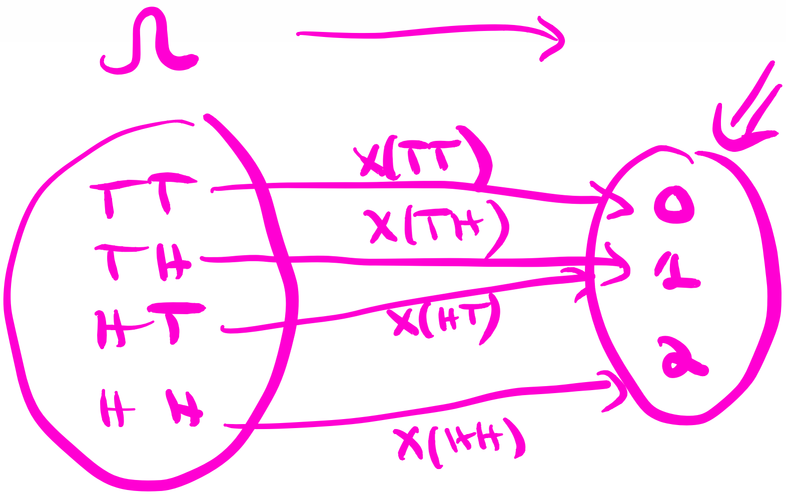
random variable

Flipping two coins

$$\Omega = \{HH, HT, TH, TT\}$$

Let X be the number of heads in two independent flips of a fair coin.

X is a function, $X: \Omega \to \mathbb{R}$ which takes outcomes $\omega \in \Omega$ and maps them to a number.



set of possible values X takes

support, range

$\Omega_X = \{0, 1, 2\}$

Random Variable

Suppose we conduct an experiment with sample space Ω. A **random variable (rv)** is a numeric function of the outcome, $X: \Omega \to \mathbb{R}$. That is, it maps outcomes $\omega \in \Omega$ to numbers, $\omega \mapsto X(\omega)$.

Random Variable

Suppose we conduct an experiment with sample space Ω. A **random variable (rv)** is a numeric function of the outcome, $X: \Omega \to \mathbb{R}$. That is, it maps outcomes $\omega \in \Omega$ to numbers, $\omega \mapsto X(\omega)$.

The set of possible values X can take on is its **range/support**, denoted Ω_X.

today's pace

a) too fast
b) about right
c) too slow