Conditional Probability LTP & Independence

slides mostly by Alex Tsun

Agenda

- conditional probability
- Bayes Theorem
- Law of Total Probability (LTP)
- Bayes Theorem + LTP

Definitions

Sample Space: The set Ω of all possible outcomes of an experiment.

- Single coin flip: $\Omega = \{H, T\}$
- Two coin flips: $\Omega = \{HH, HT, TH, TT\}$
- Roll of a die: $\Omega = \{1,2,3,4,5,6\}$

Event: Any subset $E \subseteq \Omega$.

- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
- Rolling an even number: $E = \{2,4,6\}$

$(\Omega, \Pr(\cdot))$

$\Pr(E) \to [0,1]$

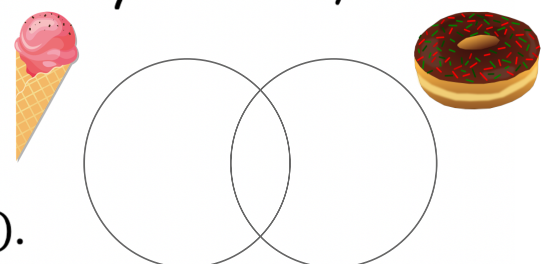
uniform prob space: $\Pr(\omega) = \frac{1}{|\Omega|}$ $\omega \in \Omega$

$\Pr(E) = \sum_{\omega \in E} \Pr(\omega) = \frac{|E|}{|\Omega|}$

Axioms of Probability & Their Consequences

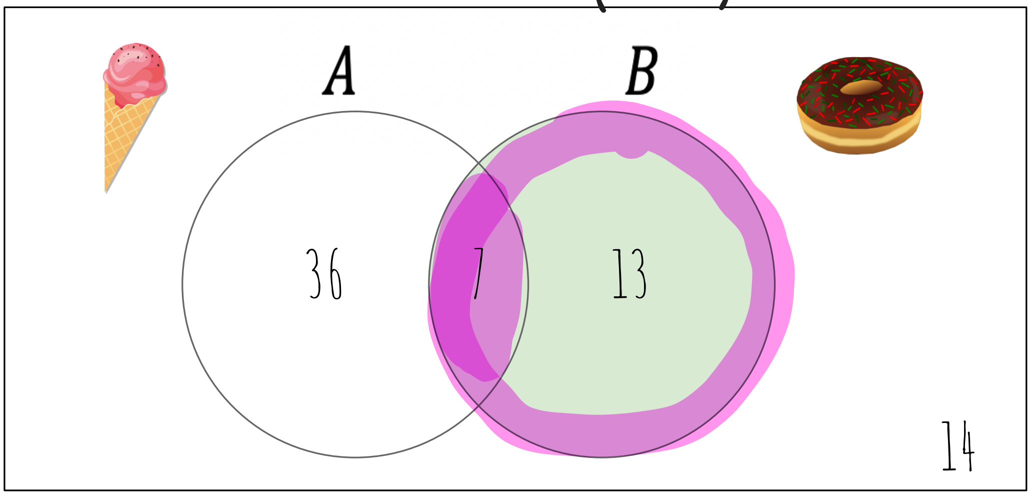
Let Ω denote the sample space and $E, F \subseteq \Omega$ be events.

Axiom 1 (Nonnegativity): $P(E) \geq 0$.
Axiom 2 (Normalization): $P(\Omega) = 1$.
Axiom 3 (Countable Additivity) If E and F are mutually exclusive, then
$P(E \cup F) = P(E) + P(F)$.



Corollary 1 (Complementation): $P(E^C) = 1 - P(E)$.
Corollary 2 (Monotonicity): If $E \subseteq F$, $P(E) \leq P(F)$.
Corollary 3 (Inclusion-Exclusion): $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

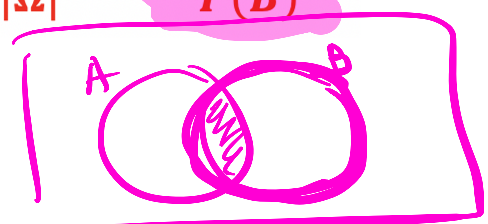
Conditional Probability (idea)



$Pr(\text{person}) = \frac{1}{70}$

What's the probability that someone likes ice cream **given** they like donuts?

$$P(A|B) = \frac{7}{20} = \frac{|A \cap B|}{|B|} = \frac{|A \cap B|/|\Omega|}{|B|/|\Omega|} = \frac{P(A \cap B)}{P(B)}$$



Conditional Probability

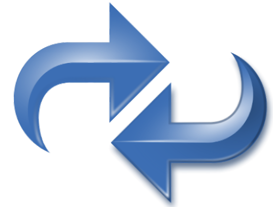
<u>**Conditional Probability**</u>: The (conditional) probability of A given an event B happened is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$Pr(B) \neq 0$

An equivalent and useful formula is $P(A \cap B) = P(A|B)P(B)$.

Conditional Probability (Reversal)



Does $P(A|B) = P(B|A)$?

Conditional Probability (Intuition)

Does $P(A|B) = P(B|A)$? No!!

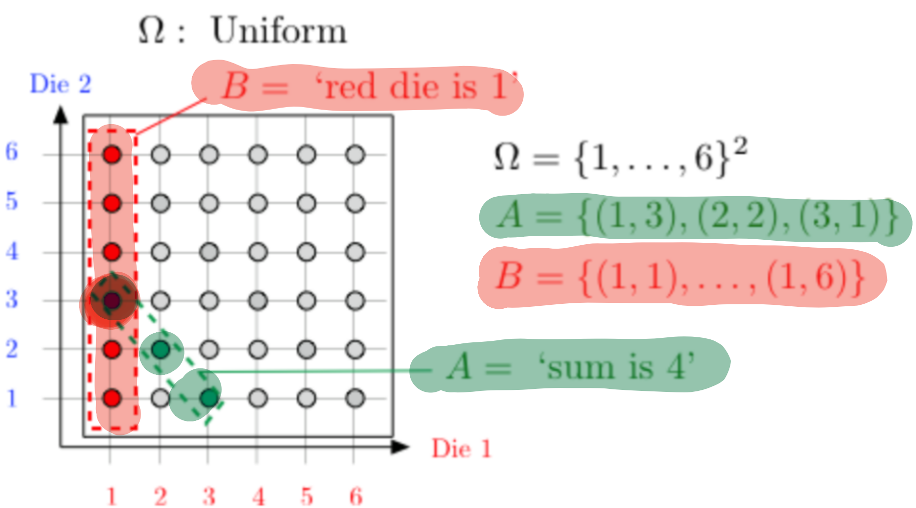
Let A be the event you are wet.
Let B be the event you are swimming.

$$P(A|B) = 1$$

$$P(B|A) \neq 1$$

Fun with conditional probability

- Toss a red die and a blue die. All outcomes equally likely. What is Pr(B | A)? What is Pr(B)?



	Pr(B)	Pr(B|A)
a)	1/6	1/6
⟹ b)	1/6	1/3
c)	1/6	3/36
d)	1/9	1/3

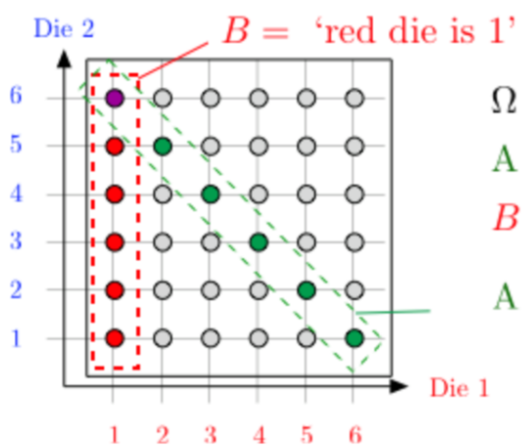
$$\Pr(B) = \frac{6}{36} = \frac{1}{6}$$

$$\Pr(B \mid A) = \frac{\Pr(B \cap A)}{\Pr(A)} = \frac{1/36}{3/36} = \frac{1}{3}$$

Fun with conditional probability

- Toss a red die and a blue die. All outcomes equally likely. What is Pr(B | A)?

Ω : Uniform



$\Omega = \{1, \ldots, 6\}^2$

$A = \{(1, 6), \ldots, (6,1)\}$

$B = \{(1, 1), \ldots, (1, 6)\}$

$A =$ 'sum is 7'

Gambler's fallacy

$\Omega = \{H, T\}^{51}$

all sequences of H/T of length 51

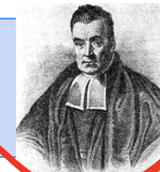
- Flip a fair coin 51 times. All outcomes equally likely.
- A = "first 50 flips are ~~heads~~" tails
- B = "the 51st flip is heads"

$Pr(\omega) = \frac{1}{2^{51}}$

- Pr (B | A) = ?

$$= \frac{Pr(B \cap A)}{Pr(A)} = \frac{1/2^{51}}{2/2^{51}} = \frac{1}{2}$$

Bayes Theorem



Bayes Theorem: Let A, B be events with nonzero probability. Then,

$$\boxed{P(A|B) = \frac{P(B|A)P(A)}{P(B)}}$$

Allows us to "reverse" the conditioning!

$P(A)$ is called the **prior** (our belief without knowing anything), and
$P(A|B)$ is called the **posterior** (our belief after learning B).

Bayes Theorem (Proof)

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

$$P(A|B)P(B) = P(A \cap B) = P(B|A)P(A)$$

$$\boxed{P(A|B) = \frac{P(B|A)P(A)}{P(B)}}$$

Bayes Theorem (Proof)

By definition of conditional probability,

$$P(A \cap B) = P(A|B)P(B)$$

Swapping A, B gives

$$P(B \cap A) = P(B|A)P(A)$$

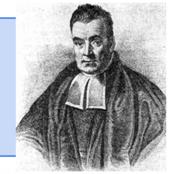
But $P(A \cap B) = P(B \cap A)$, so

$$P(A|B)P(B) = P(B|A)P(A)$$

Dividing both sides by $P(B)$ gives

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Bayes Theorem



Bayes Theorem: Let A, B be events with nonzero probability. Then,

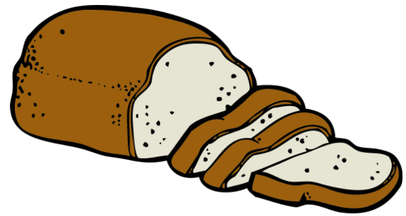
$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Allows us to "reverse" the conditioning!

$P(A)$ is called the **prior** (our belief without knowing anything), and
$P(A|B)$ is called the **posterior** (our belief after learning B).

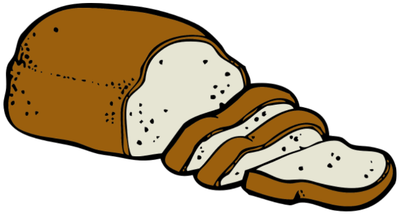
Random Picture

Law of Total Probability



Cutting up a Sample Space

Ω

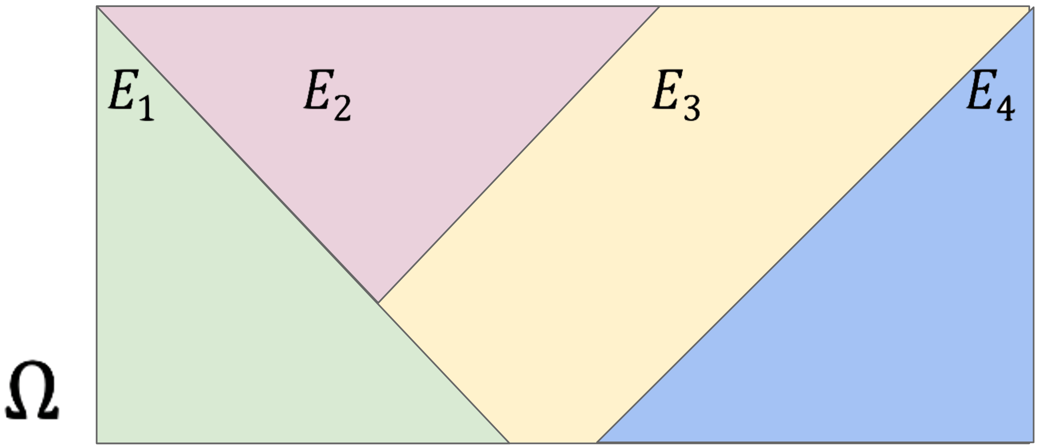


Cutting up a Sample Space

These events **partition** the sample space i.e.,

1. They "cover" the whole space.

2. They don't overlap.

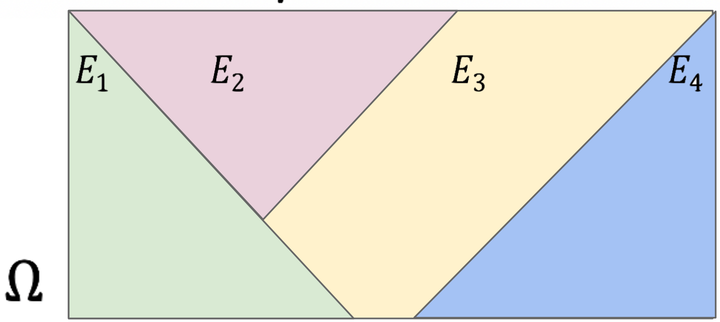


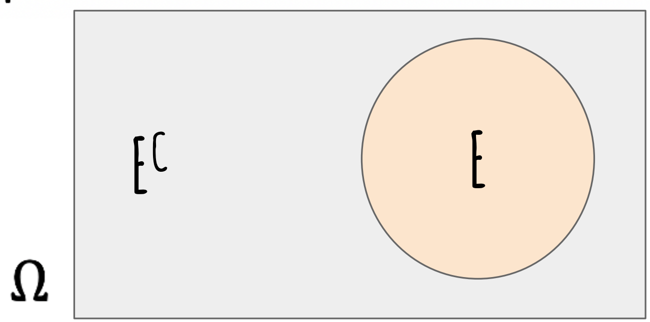
Partitions

Partition: Non-empty events $E_1, \ldots, E_n$ partition the sample space Ω if

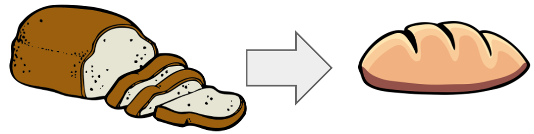
- (**Exhaustive**) $E_1 \cup E_2 \cup \ldots \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega$.
- (**Pairwise Mutually Exclusive**) For all $i \neq j$, $E_i \cap E_j = \emptyset$.

Notice for any event E: E and E^C always partition Ω.



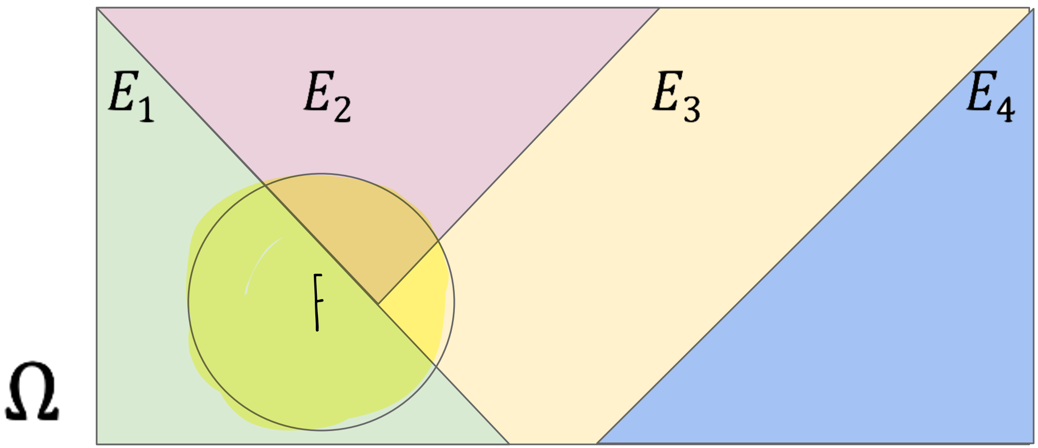


(The Picture) Law of total probability

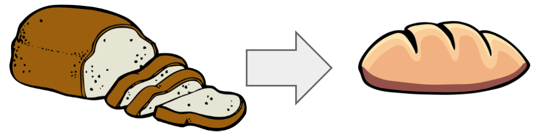


Back to the old picture. How can we decompose event F?

$$P(F) = P(\underline{E_1 \cap F}) + P(\underline{E_2 \cap F}) + P(\underline{E_3 \cap F}) + \underbrace{\cancel{P(\underline{E_4 \cap F})}}_{0}$$

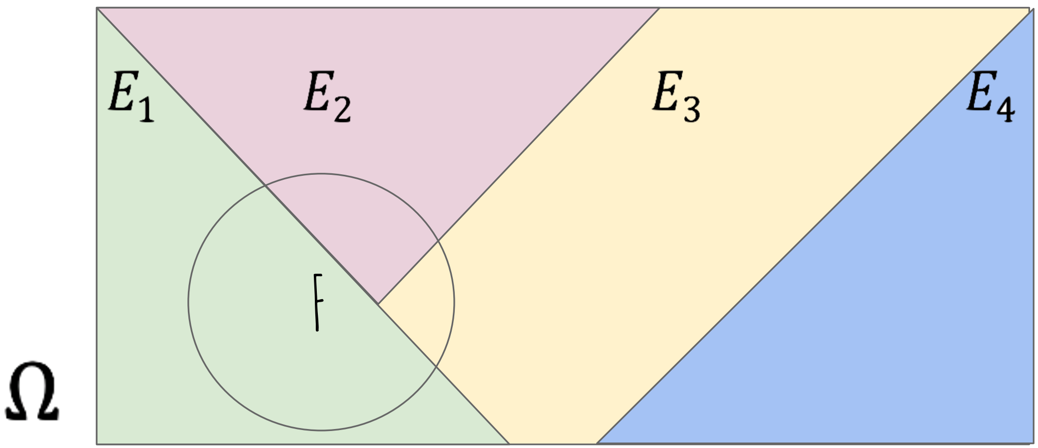


(The Picture) Law of total probability

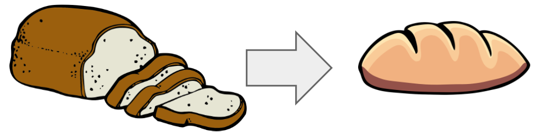


Back to the old picture. How can we decompose event **F**?

$$P(F) = P(F \cap E_1) + P(F \cap E_2) + P(F \cap E_3)$$

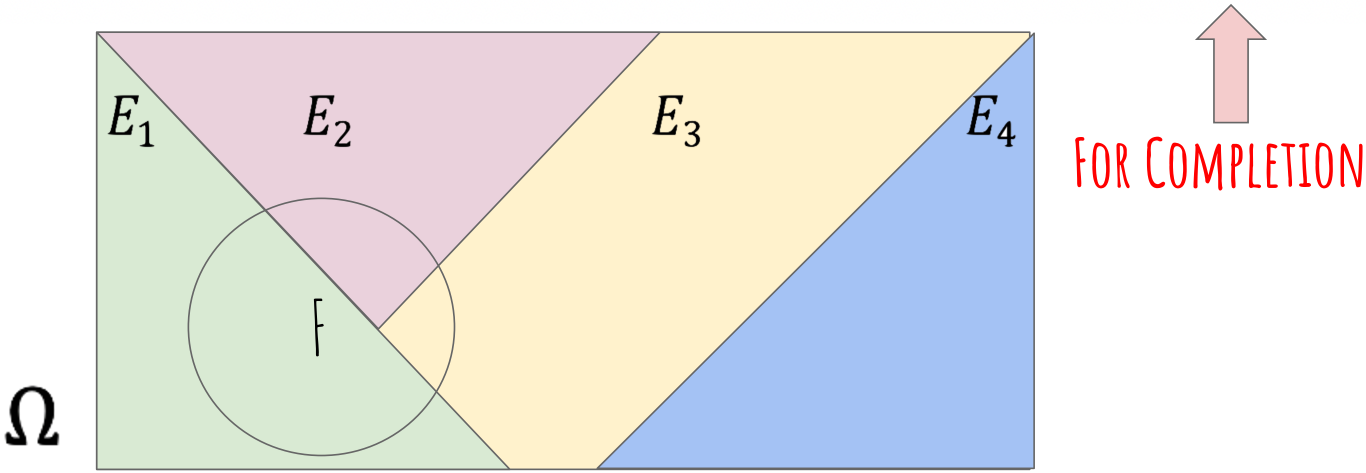


(The Picture) Law of total probability



Back to the old picture. How can we decompose event **F**?

$$P(F) = P(F \cap E_1) + P(F \cap E_2) + P(F \cap E_3) + P(F \cap E_4)$$



Law of Total Probability (LTP)

Law of Total Probability: If events $E_1, \dots, E_n$ partition Ω, then for any event F,

$$P(F) = P(F \cap E_1) + \cdots + P(F \cap E_n) = \sum_{i=1}^{n} P(F \cap E_i)$$

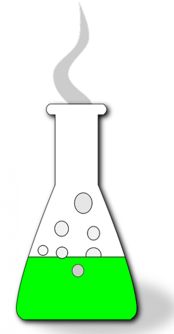
Using the definition of conditional probability ($P(F \cap E_i) = P(F|E_i)P(E_i)$), we get an alternate (more useful) form

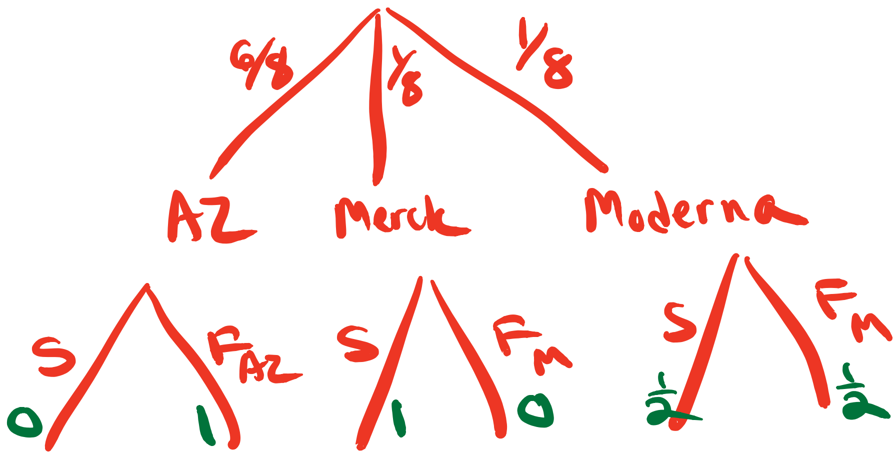
$$P(F) = P(F|E_1)P(E_1) + \cdots + P(F|E_n)P(E_n) = \sum_{i=1}^{n} P(F|E_i)P(E_i)$$

$$P(F|E_1) = \frac{P(F \cap E_1)}{P(E_1)}$$

Intuition (LTP)

$$P(F) = P(F|E_1)P(E_1) + \cdots + P(F|E_n)P(E_n) = \sum_{i=1}^{n} P(F|E_i)P(E_i)$$



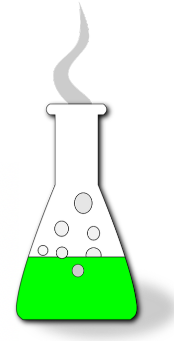


Pr(investment fails to pan out)

F

Intuition (LTP)

$$P(F) = P(F|E_1)P(E_1) + \cdots + P(F|E_n)P(E_n) = \sum_{i=1}^{n} P(F|E_i)P(E_i)$$



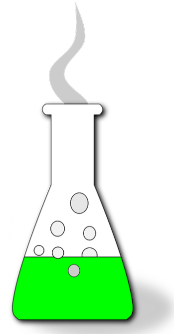
- You want to know the probability company you invested in fails to produce a successful vaccine
- You chose randomly which company to invest in

    First, compute the probability of failure for each of 3 companies

    Then, weight those by the probability of investing in that company

Example (LTP)

$$P(F) = P(F|E_1)P(E_1) + \cdots + P(F|E_n)P(E_n) = \sum_{i=1}^{n} P(F|E_i)P(E_i)$$



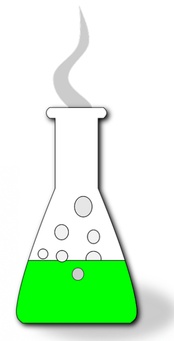
	AstraZeneca (E_1)	Merck (E_2)	Moderna (E_3)
Probability invest in this company	6/8	1/8	1/8
Probability this company's vaccine fails to work	1	0	1/2

$$Pr(F) = Pr(F|E_1)Pr(E_1) + Pr(F|E_2)Pr(E_2) + Pr(F|E_3)Pr(E_3)$$

$$1 \cdot \frac{6}{8} + 0 \cdot \frac{1}{8} + \frac{1}{2} \cdot \frac{1}{8}$$

Example (LTP)

$$P(F) = P(F|E_1)P(E_1) + \cdots + P(F|E_n)P(E_n) = \sum_{i=1}^{n} P(F|E_i)P(E_i)$$

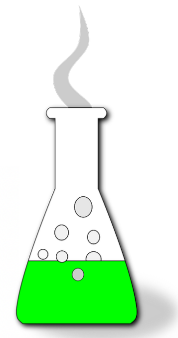


	AstraZeneca (E_1)	Merck (E_2)	Moderna (E_3)
Probability invest in this company	$6/8$	$1/8$	$1/8$
Probability this company's vaccine fails to work	1	0	$1/2$

$$P(F) = P(F|E_1)P(E_1) + P(F|E_2)P(E_2) + P(F|E_3)P(E_3) = 1 \cdot \frac{6}{8} + 0 \cdot \frac{1}{8} + \frac{1}{2} \cdot \frac{1}{8} = \mathbf{\frac{13}{16}}$$

Example (LTP)

$$P(F) = P(F|E_1)P(E_1) + \cdots + P(F|E_n)P(E_n) = \sum_{i=1}^{n} P(F|E_i)P(E_i)$$



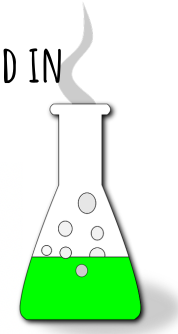
	AstraZeneca (E_1)	Merck (E_2)	Moderna (E_3)
Probability invest in this company	6/8	1/8	1/8
Probability this company's vaccine fails to work	1	0	1/2

What is the probability invested in Moderna given investment did not pay off (vaccine of company you invested in fails to work)?

$Pr(E_3 | F)$

WHAT'S THE PROBABILITY THAT YOU INVESTED IN MODERNA, GIVEN THAT THE COMPANY YOU INVESTED IN FAILS TO PRODUCE SUCCESSFUL VACCINE?

NEED LTP FOR DENOMINATOR...



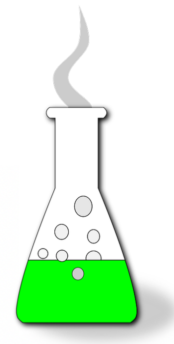
$$P(F) = P(F|E_1)P(E_1) + \cdots + P(F|E_n)P(E_n) = \sum_{i=1}^{n} P(F|E_i)P(E_i)$$

		Moderna (E_3)
Probability invest in this company	$P(E_3|F) = \frac{P(F|E_3)P(E_3)}{P(F)} = \frac{\frac{1}{2}\cdot\frac{1}{8}}{13/16} = 1/13$	$1/8$
Probability this company's vaccine fails to work		$1/2$

$$P(F) = P(F|E_1)P(E_1) + P(F|E_2)P(E_2) + P(F|E_3)P(E_3) = 1\cdot\frac{6}{8} + 0\cdot\frac{1}{8} + \frac{1}{2}\cdot\frac{1}{8} = \frac{13}{16}$$

Example (LTP)

$$P(F) = P(F|E_1)P(E_1) + \cdots + P(F|E_n)P(E_n) = \sum_{i=1}^{n} P(F|E_i)P(E_i)$$



		Moderna (E_3)
Probability invest in this company	$P(E_3|F) = \frac{P(F|E_3)P(E_3)}{P(F)} = \frac{\frac{1}{2}\cdot\frac{1}{8}}{\frac{13}{16}} = \frac{1}{13}$	$1/8$
Probability this company's vaccine fails to work		$1/2$

$$P(F) = P(F|E_1)P(E_1) + P(F|E_2)P(E_2) + P(F|E_3)P(E_3) = 1\cdot\frac{6}{8} + 0\cdot\frac{1}{8} + \frac{1}{2}\cdot\frac{1}{8} = \frac{13}{16}$$

Bayes Theorem with Law of Total Probability

Bayes Theorem with LTP: Let $E_1, \ldots, E_n$ be a partition of the sample space, and F an event. Then,

$$P(E_1|F) = \frac{P(F|E_1)P(E_1)}{P(F)} = \frac{P(F|E_1)P(E_1)}{\sum_{i=1}^{n} P(F|E_i)P(E_i)}$$

(Simple Partition) In particular, if E is an event with nonzero probability, then

$$P(E|F) = \frac{P(F|E)P(E)}{P(F)} = \frac{P(F|E)P(E)}{P(F|E)P(E) + P(F|E^C)P(E^C)}$$

Another Example

$Pr(\bar{G}) = 1 - P(G)$

$Pr(G) = 0.01$

- 1% of people have a certain genetic disorder.
- 90% of time people with disorder test positive (true positives)

  $Pr(\text{test pos} \mid G) = 0.9$
- 9.6% of the time, people that don't have the disorder test ~~negative~~ positive (false positives).

  $Pr(\text{test pos} \mid \bar{G}) = 0.096$
- If a person gets a positive test result, what is the probability that they actually have the disorder?

G: have the disorder

~~P~~ts: positive test result

$$\boxed{Pr(G \mid \text{test pos})} = \frac{\overset{0.9}{Pr(\text{test pos} \mid G)}\ \overset{0.01}{P(G)}}{P(\text{test pos})}$$

$$P(\text{test pos}) = Pr(\text{test pos} \mid G)P(G) + P(\text{test pos} \mid \overline{G})P(\overline{G})$$

$$0.9 \quad 0.01 \quad 0.096 \quad 0.99$$

ANOTHER EXAMPLE

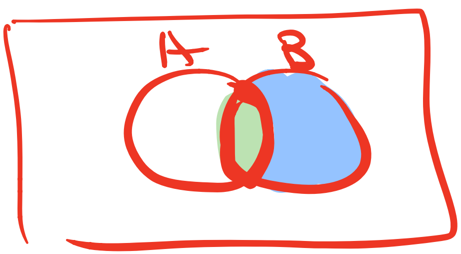
- 1% of people have a certain genetic disorder.
- 90% of time people with disorder test positive (true positives)
- 9.6% of the time, people that don't have the disorder test ~~negative~~ positive (false positives).
- If a person gets a positive test result, what is the probability that they actually have the disorder?

G: have the disorder
P: positive test result

$$Pr(\text{test pos} \mid G) = 0.9$$

$$Pr(\text{test neg} \mid G) = 0.1$$

$$= 1 - Pr(\text{test pos} \mid G)$$



$$Pr(A \mid B) + Pr(\overline{A} \mid B) = \frac{Pr(A \cap B)}{Pr(B)} + \frac{Pr(\overline{A} \cap B)}{P(B)}$$

$$= \frac{P(B)}{P(B)} = 1$$

Probability

2.3 Independence

Most slides by Alex Tsun

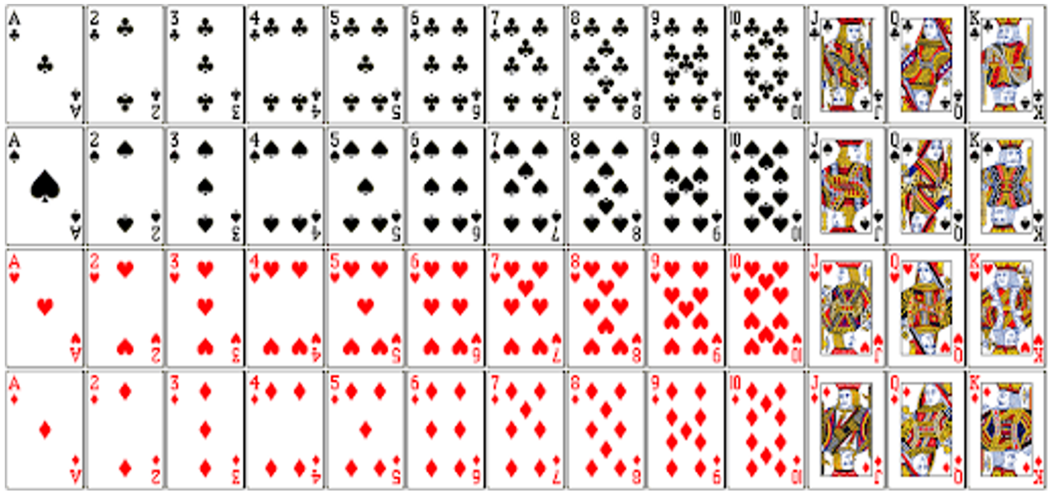
Agenda

- Chain Rule
- Independence
- Conditional Independence

Chain Rule (Idea)

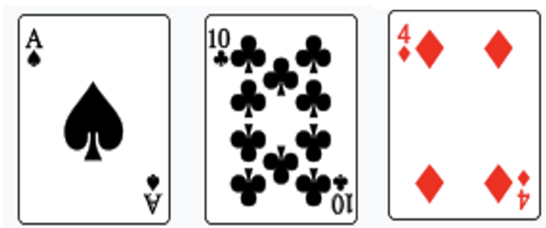
Have a Standard 52-Card Deck.

- 4 Suits (Clubs, Diamonds, Hearts, Spades)
- 13 ranks (A, 2, 3, ..., 9, 10, J, Q, K)



Chain Rule (Idea)

Have a Standard 52-Card Deck. Shuffle It, and draw the top 3 cards.

What is P () = P(**A**, **B**, **C**)?

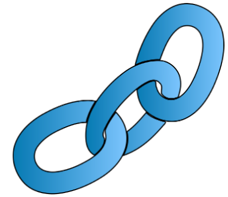
$\frac{1}{52} \cdot \frac{1}{51} \cdot \frac{1}{50}$

A: Ace of Spades First

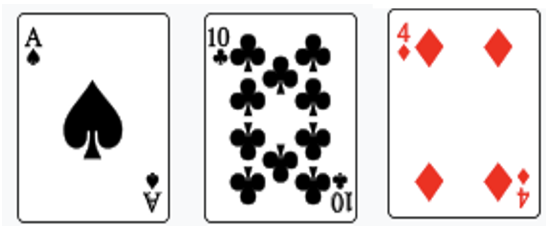
B: 10 of Clubs Second

C: 4 of Diamonds Third

Chain Rule (Idea)



Have a Standard 52-Card Deck. Shuffle It, and draw the top 3 cards.

What is P () = P(**A**, **B**, **C**)?

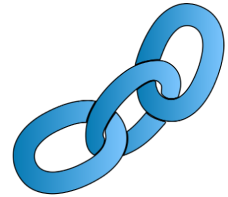
$$\frac{1}{52} \cdot \frac{1}{51} \cdot \frac{1}{50}$$

A: Ace of Spades First

B: 10 of Clubs Second

C: 4 of Diamonds Third

Chain Rule (Idea)

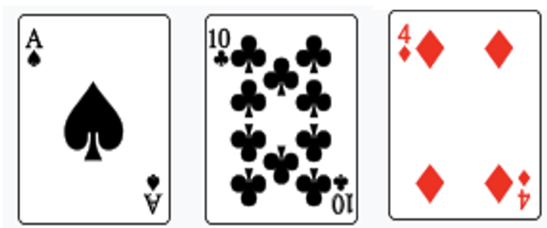


Have a Standard 52-Card Deck. Shuffle It, and draw the top 3 cards. (uniform probability space).

A: Ace of Spades First

B: 10 of Clubs Second

C: 4 of Diamonds Third

What is P () = P(A, B, C)?

$$\frac{1}{52} \cdot \frac{1}{51} \cdot \frac{1}{50}$$

$$P(A) \cdot P(B|A) \cdot P(C|A,B)$$

Chain Rule

Chain Rule: Let $A_1, \dots, A_n$ be events with nonzero probability. Then,

$$P(A_1, \dots, A_n) = P(A_1)P(A_2|A_1)P(A_3|\underbrace{A_1, A_2}_{A_1 \cap A_2}) \dots P(A_n|\underbrace{A_1, \dots, A_{n-1}}_{A_1 \cap \dots \cap A_{n-1}})$$

In the case of two events A, B,

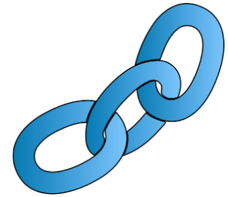
$$P(A, B) = P(A)P(B|A)$$

An easy way to remember this formula: we need to do n tasks, so we can perform them one at a time, conditioning on what we've done so far.

$$\Pr(A \cap B \cap C) \overset{?}{=} P(A)\,P(B|A)\,P(C|A \cap B)$$

$$= \cancel{P(A)} \cdot \frac{\cancel{P(B \cap A)}}{\cancel{P(A)}} \cdot \frac{P(C \cap A \cap B)}{\cancel{P(A \cap B)}}$$

Chain Rule (Idea)

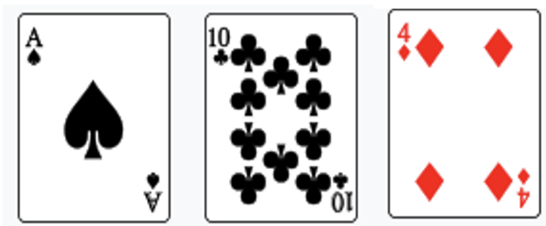


Have a standard 52-card deck. Shuffle it, and draw the top 3 cards. (uniform probability space).

A: Ace of Spades First

B: 10 of Clubs Second

C: 4 of Diamonds Third

What is P() = P(A, B, C)?

$$\frac{1}{52} \cdot \frac{1}{51} \cdot \frac{1}{50}$$

$$P(A) \cdot P(B|A) \cdot P(C|A,B)$$

a) $\frac{1}{51}$

b) $\frac{1}{52}$

c) $\frac{1}{13}$

d) $\frac{1}{4}$

Fun with cards

- Two people, A and B, are playing the following game.
- A 6-sided die is thrown and each time it’s thrown, regardless of the history, it is equally likely to show any of the six numbers
- If it shows 5, A wins.
- If it shows 1, 2 or 6, B wins.
- Otherwise, they play a second round and so on.

- What is Pr(A wins on 4^{th} round)?

Fun with cards

Two people, A and B, are playing the following game.
A 6-sided die is thrown and each time it’s thrown, regardless of the history, it is equally likely to show any of the six numbers
If it shows 5, A wins.
If it shows 1, 2 or 6, B wins.
Otherwise, they play a second round and so on.

What is Pr(A wins on 4^{th} round)?

The need for independence

Quick question: In *general*, is

$$P(A, B) = P(A)P(B)?$$

The need for independence

Quick question: In *general*, is

$$P(A, B) = P(A)P(B)?$$

The chain rule says

$$P(A, B) = P(A)P(B|A)$$

So no, unless the special case when $P(B|A) = P(B)$. This case is so important it has a name.

Independence

Independence: Events A, B are independent if any of the three equivalent conditions hold:

1. $P(A|B) = P(A)$
2. $P(B|A) = P(B)$
3. $P(A, B) = P(A)P(B)$

Independence

- Toss a coin 3 times. Each of 8 outcomes equally likely.

Define

- A = {at most one T} = {HHH, HHT, HTH, THH}
- B = {at most 2 Heads}= $\{HHH\}^c$

- Are A and B independent?

	$Pr(A)$	$Pr(B)$	$Pr(A \cap B)$
a)	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{16}$
b)	$\frac{1}{2}$	$\frac{7}{8}$	$\frac{7}{16}$
c)	$\frac{1}{2}$	$\frac{7}{8}$	$\frac{3}{8}$
d)	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{21}{32}$

Using independence to define a probabilistic model

- We can **define** our probability model via independence.
- Example: suppose a biased coin comes up heads with probability 2/3, independent of other flips.
- Sample space: sequences of 3 coin tosses.

- Pr (HHH)=?
- Pr (TTT) = ?
- Pr (HHT) = ?
- Pr (2 heads) = ?