2. Probability

Intro to Discrete Probability

Anna Karlin

With many slides by Alex Tsun and CS70 at berkeley

Agenda

- Definitions
- Axioms
- Equally Likely Outcomes
- Beyond equally likely outcomes
- Conditional Probability

Definitions

Sample Space: The set Ω of all possible outcomes of an experiment.

- Single coin flip: $\Omega = \{H, T\}$
- Two coin flips: $\Omega = \{HH, HT, TH, TT\}$
- Roll of a die: $\Omega = \{1,2,3,4,5,6\}$

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Event: Any subset $E \subseteq \Omega$.

- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
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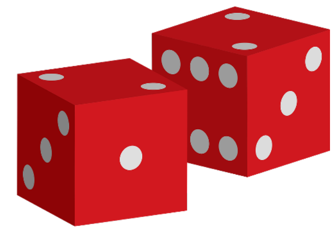
Mutually Exclusive: Events E and F are mutually exclusive if $E \cap F = \emptyset$ (i.e., they can't simultaneously happen).

- $E = \{2,4,6\}$ and $F = \{1,3\}$, then $E \cap F = \emptyset$.

Example: weird dice (Sample Space)



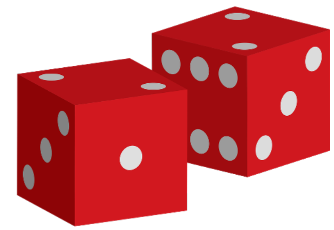
Suppose I roll two 4-sided dice. Here is the sample space (set of possible outcomes)

Die 2 (red)

Die 1 (blue)

	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

Example: Weird Dice (Events)



Let D1 be the value of the blue die, and D2 the value of the red die.

What outcomes match these events?

A. D1 = 1

B. D1 + D2 = 6

C. D1 = 2 * D2

Die 2 (red)

Die 1 (blue)

	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
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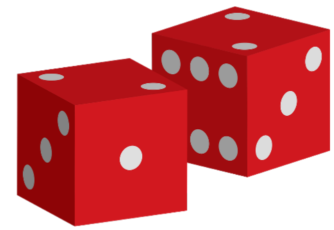


Are A and B mutually exclusive?

Are B and C mutually exclusive?

A. D1 = 1

B. D1 + D2 = 6

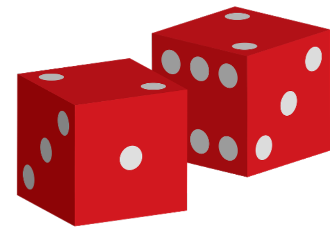
C. D1 = 2 * D2

Die 2 (Red)

Die 1 (Blue)

Die 1 (Blue) \ Die 2 (Red)	1	2	3	4
1	(1, 1) [A]	(1, 2) [A]	(1, 3) [A]	(1, 4) [A]
2	(2, 1) [C]	(2, 2)	(2, 3)	(2, 4) [B]
3	(3, 1)	(3, 2)	(3, 3) [B]	(3, 4)
4	(4, 1)	(4, 2) [B, C]	(4, 3)	(4, 4)

Example: weird dice (mutually exclusive)



Are A and B mutually exclusive?

Yes. $A \cap B = \emptyset$ (no overlap)

Die 2 (red)

Die 1 (blue)

	1	2	3	4
1	(1, 1) A	(1, 2) A	(1, 3) A	(1, 4) A
2	(2, 1) C	(2, 2)	(2, 3)	(2, 4) B
3	(3, 1)	(3, 2)	(3, 3) B	(3, 4)
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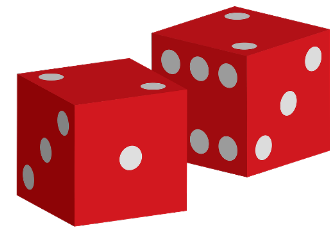
Example: Weird Dice (Mutually Exclusive)



Are B and C mutually exclusive?

No. B and C could happen at the same time (4, 2)

Die 2 (Red)

Die 1 (Blue)

Die 1 (Blue) \ Die 2 (Red)	1	2	3	4
1	(1, 1) A	(1, 2) A	(1, 3) A	(1, 4) A
2	(2, 1) C	(2, 2)	(2, 3)	(2, 4) B
3	(3, 1)	(3, 2)	(3, 3) B	(3, 4)
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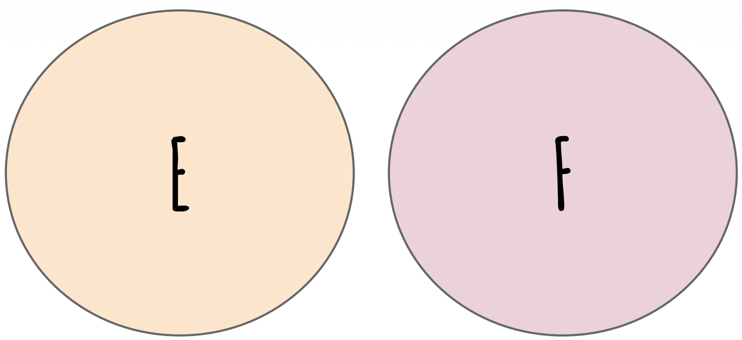
Random Picture

Axioms of Probability & Their Consequences

Let Ω denote the sample space and $E, F \subseteq \Omega$ be events.

Axiom 1 (Nonnegativity): $P(E) \geq 0$.
Axiom 2 (Normalization): $P(\Omega) = 1$.
Axiom 3 (Countable Additivity) If E and F are mutually exclusive, then $P(E \cup F) = P(E) + P(F)$.



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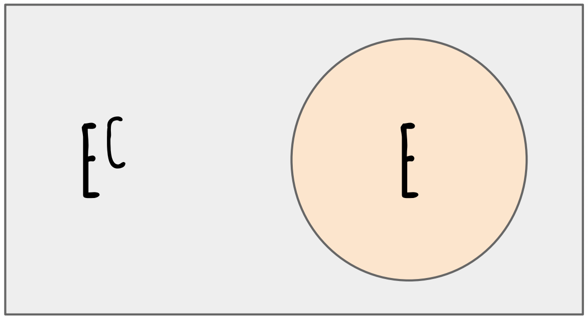
Corollary 1 (Complementation): $P(E^C) = 1 - P(E)$.

Ω



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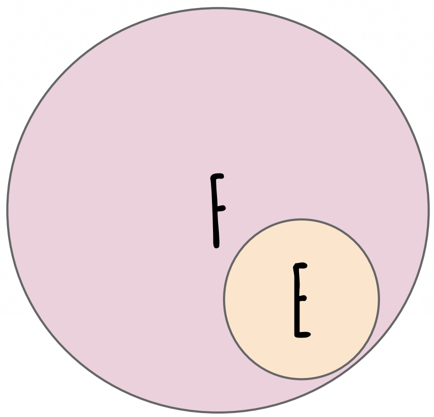

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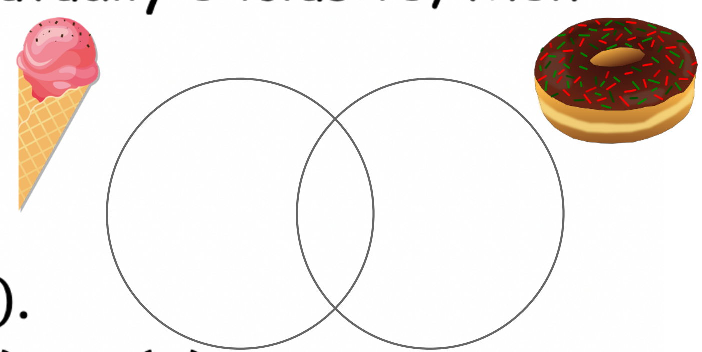

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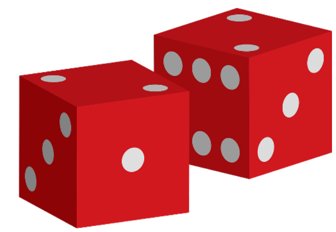

Think back to the 4-sided dice. Suppose each die is fair.

Intuitively, what is the probability that the two dice sum to 6? (D1 + D2 = 6)

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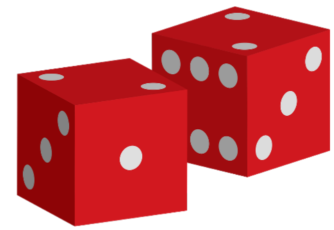

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Die 1 (blue)

Each of the 16 outcomes is equally likely.

3/16.

Equally Likely Outcomes

If Ω is such that outcomes are equally likely, then for any event $E \subseteq \Omega$,

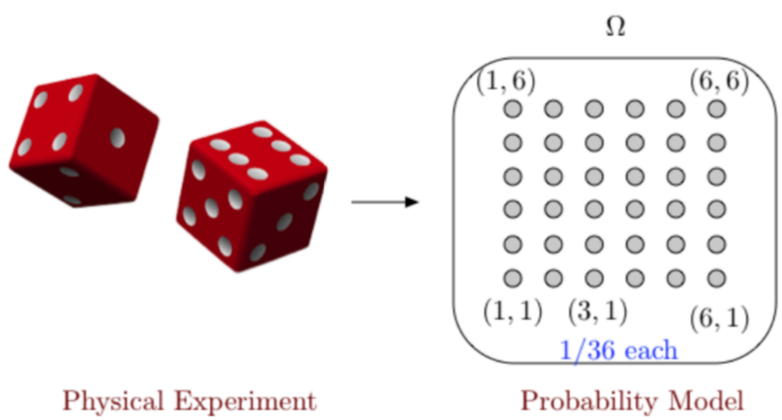
$$P(E) = \frac{|E|}{|\Omega|}$$

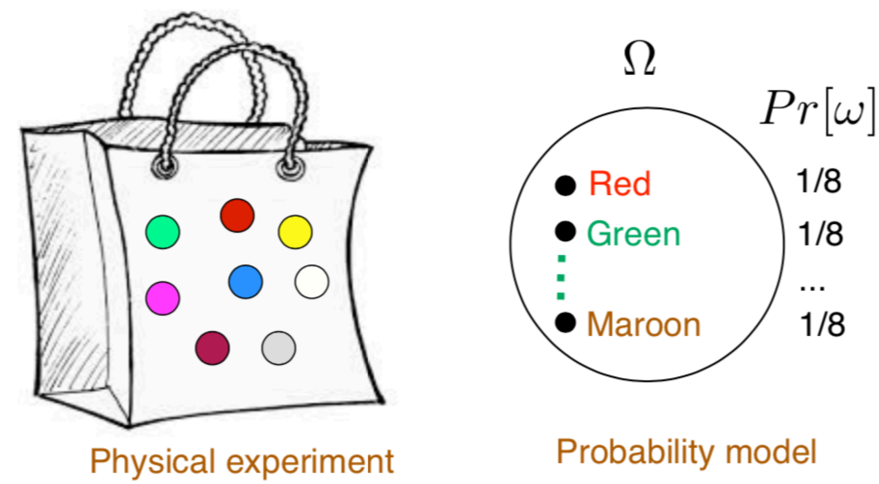
Coin tossing

Toss a coin 100 times. Each outcome is equally likely. What is the probability of seeing 50 heads?

Non-equally Likely outcomes

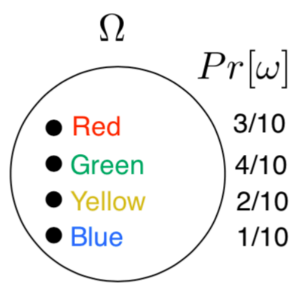
More examples – uniform probability spaces



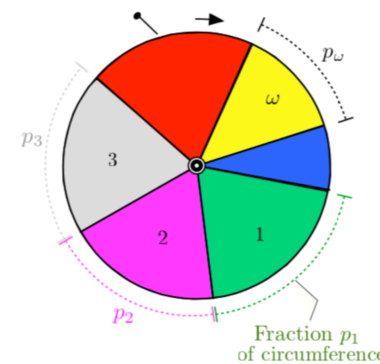


Nonuniform probability spaces

Physical experiment



Probability model



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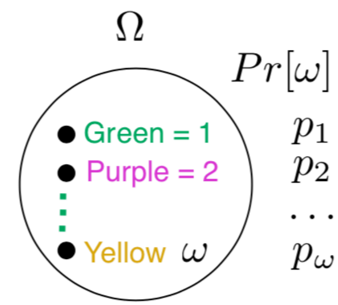

Probability model

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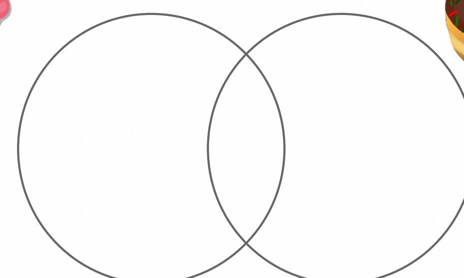

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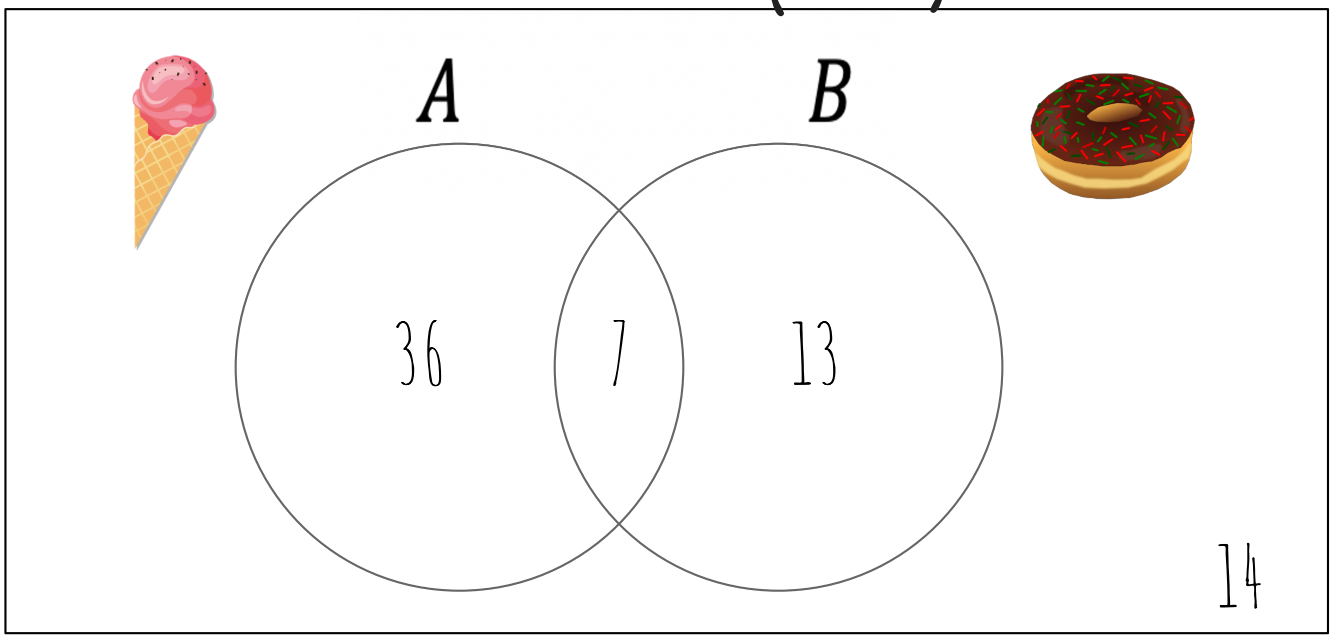
Conditional Probability

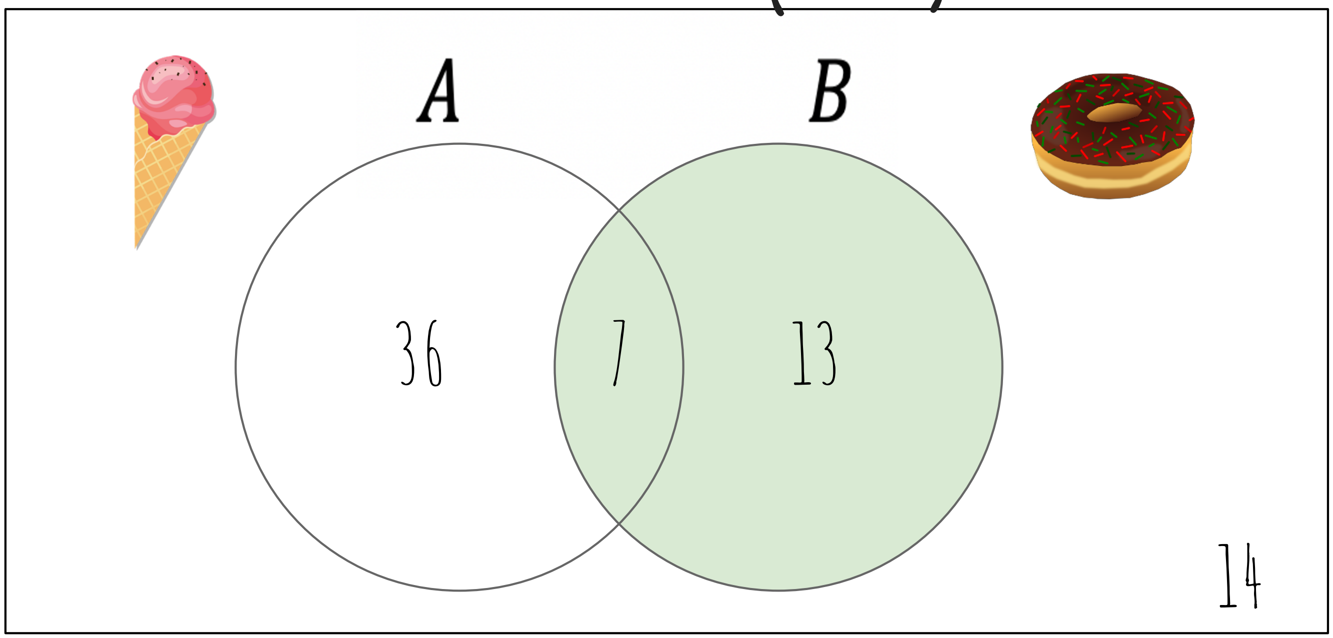
slides mostly by Alex Tsun

Conditional Probability (idea)



What's the probability that someone likes ice cream **given** they like donuts?

Conditional Probability (idea)



What's the probability that someone likes ice cream **given** they like donuts?

$$P(A|B) = \frac{7}{20} = \frac{|A \cap B|}{|B|} = \frac{|A \cap B|/|\Omega|}{|B|/|\Omega|} = \frac{P(A \cap B)}{P(B)}$$

Conditional Probability

Conditional Probability: The (conditional) probability of A given an event B happened is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

An equivalent and useful formula is $P(A \cap B) = P(A|B)P(B)$.

Conditional Probability (Reversal)

Does $P(A|B) = P(B|A)$?

Conditional Probability (intuition)

Does $P(A|B) = P(B|A)$? No!!

Let A be the event you are wet.
Let B be the event you are swimming.

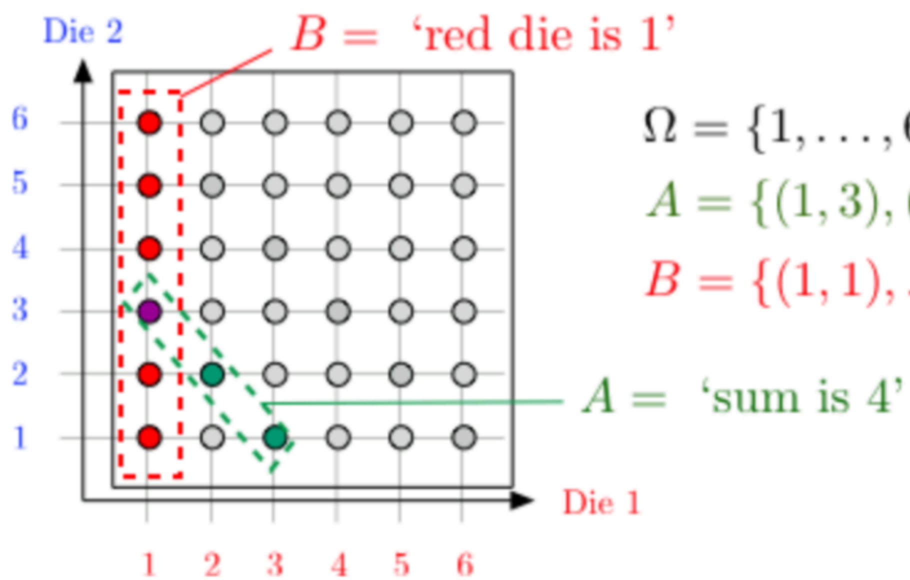
$$P(A|B) = 1$$

$$P(B|A) \neq 1$$

Fun with conditional probability

- Toss a red die and a blue die. All outcomes equally likely. What is Pr(B | A)? What is Pr(B)?

Ω : Uniform



$$\Omega = \{1, \ldots, 6\}^2$$
$$A = \{(1,3), (2,2), (3,1)\}$$
$$B = \{(1,1), \ldots, (1,6)\}$$

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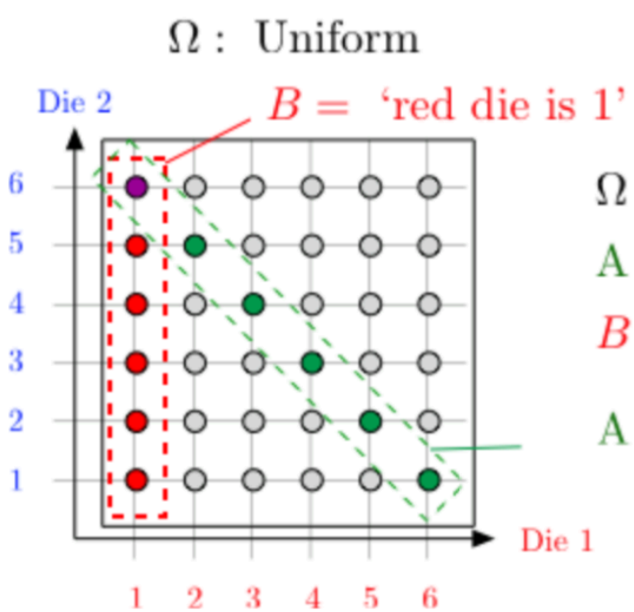

$\Omega = \{1, \ldots, 6\}^2$

$A = \{(1, 6), \ldots, (6,1)\}$

$B = \{(1, 1), \ldots, (1, 6)\}$

A = 'sum is 7'

Gambler's fallacy

- Flip a fair coin 51 times. All outcomes equally likely.
- A = "first 50 flips are heads"
- B = "the 51^{st} flip is heads"

- Pr (B | A) = ?