Announcements
- pset 2 posted after lecture
- partner finding form due by noon.

2. Probability

Intro to Discrete Probability

Anna Karlin

With many slides by Alex Tsun and CS70 at Berkeley

Agenda

- Definitions
- Axioms
- Equally Likely Outcomes
- Beyond equally likely outcomes
- Conditional Probability

way to model uncertain events

$\boxed{\text{random experiment}} \rightarrow$ outcome.

DEFINITIONS

Sample Space: The set Ω of all possible outcomes of an experiment.

- Single coin flip: $\Omega = \{H, T\}$
- Two coin flips: $\Omega = \{HH, HT, TH, TT\}$
- Roll of a die: $\Omega = \{1,2,3,4,5,6\}$

probability space

assigns number $P(\omega)$

to each $\omega \in \Omega$

Definitions

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Event: Any subset $E \subseteq \Omega$.

- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
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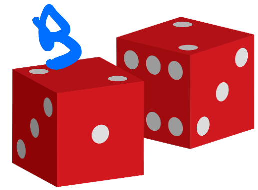
- Rolling an even number: $E = \{2,4,6\}$

Mutually Exclusive: Events E and F are mutually exclusive if $E \cap F = \emptyset$ (i.e., they can't simultaneously happen).

- $E = \{2,4,6\}$ and $F = \{1,3\}$, then $E \cap F = \emptyset$.

Example: weird dice (Sample Space)

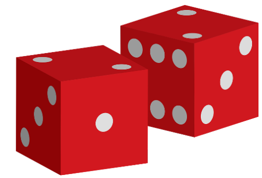


Suppose I roll two 4-sided dice. Here is the sample space (set of possible outcomes)

Die 2 (red)

Die 1 (blue)

	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)



Example: Weird Dice (Events)

Let D1 be the value of the blue die, and D2 the value of the red die.

What outcomes match these events?

A. D1 = 1

B. D1 + D2 = 6

C. D1 = 2 * D2

Die 2 (Red)

Die 1 (Blue)

Die 1 (Blue) \ Die 2 (Red)	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
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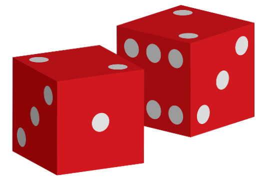
Example: Weird Dice (Events)



Are A and B mutually exclusive?

Are B and C mutually exclusive?

A. $D1 = 1$

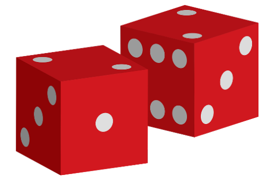
B. $D1 + D2 = 6$

C. $D1 = 2 * D2$

Die 2 (Red)

Die 1 (Blue)

	1	2	3	4
1	(1, 1) A	(1, 2) A	(1, 3) A	(1, 4) A
2	(2, 1) C	(2, 2)	(2, 3)	(2, 4) B
3	(3, 1)	(3, 2)	(3, 3) B	(3, 4)
4	(4, 1)	(4, 2) B C	(4, 3)	(4, 4)



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Die 1 (Blue)

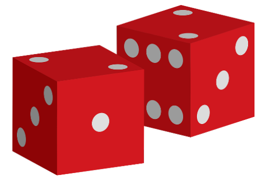
	1	2	3	4
1	(1, 1) A	(1, 2) A	(1, 3) A	(1, 4) A
2	(2, 1) C	(2, 2)	(2, 3)	(2, 4) B
3	(3, 1)	(3, 2)	(3, 3) B	(3, 4)
4	(4, 1)	(4, 2) B C	(4, 3)	(4, 4)

	A & B	B & C
(a)	yes	yes
(b)	yes	no
(c)	no	yes

(c) no yes

(d) no no

Example: weird dice (mutually exclusive)



Are A and B mutually exclusive?

Yes. $A \cap B = \emptyset$ (no overlap)

Die 2 (red)

Die 1 (blue)

	1	2	3	4
1	(1, 1) A	(1, 2) A	(1, 3) A	(1, 4) A
2	(2, 1) C	(2, 2)	(2, 3)	(2, 4) B
3	(3, 1)	(3, 2)	(3, 3) B	(3, 4)
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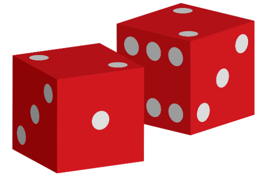
Example: weird dice (mutually exclusive)



Are B and C mutually exclusive?

No. B and C could happen at the same time (4, 2)

Die 2 (red)

Die 1 (blue)

	1	2	3	4
1	(1, 1) A	(1, 2) A	(1, 3) A	(1, 4) A
2	(2, 1) C	(2, 2)	(2, 3)	(2, 4) B
3	(3, 1)	(3, 2)	(3, 3) B	(3, 4)
4	(4, 1)	(4, 2) B C	(4, 3)	(4, 4)

Random Picture

Axioms of Probability & Their Consequences

Let Ω denote the sample space and $E, F \subseteq \Omega$ be events.

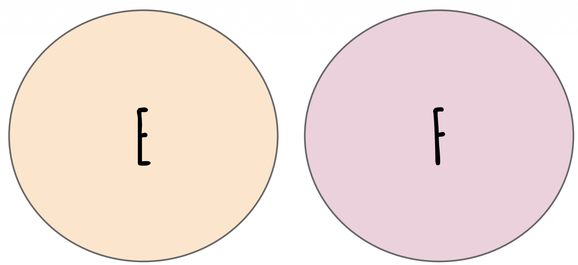
Axiom 1 (Nonnegativity): $P(E) \geq 0$.

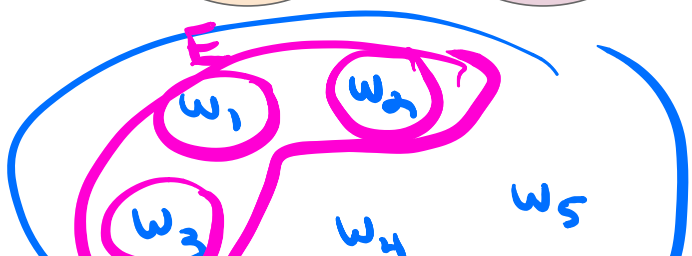
Axiom 2 (Normalization): $P(\Omega) = 1$.

Axiom 3 (Countable Additivity) If E and F are mutually exclusive, then $P(E \cup F) = P(E) + P(F)$.

$$\boxed{\Pr(E) = \sum_{w \in E} \Pr(w)}$$

(1,2)





Axioms of Probability & Their Consequences

Let Ω denote the sample space and $E, F \subseteq \Omega$ be events.

$P : E \to \mathbb{R}_{\geq 0}$

$[0,1]$

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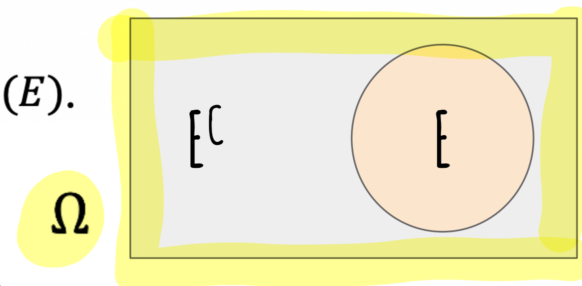
Corollary 1 (Complementation): $P(E^C) = 1 - P(E)$.



$\Omega = E \cup E^c$

$1 = \Pr(\Omega) = \Pr(E) + \Pr(E^c)$

Axioms of Probability & Their Consequences

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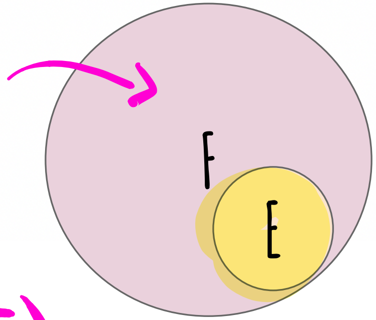

$$P(F) = P(E) + \underbrace{P(F \setminus E)}_{\geq 0}$$

$$\Rightarrow P(F) \geq P(E)$$

Axioms of Probability & Their Consequences

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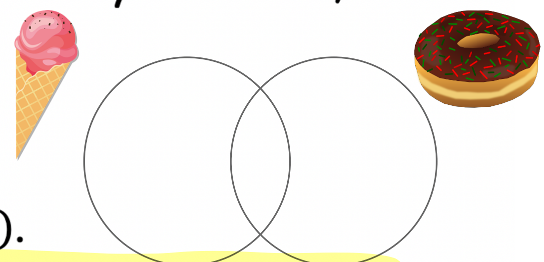

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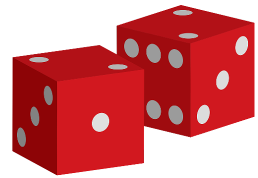
Example: Weird Dice (Events)



Think back to the 4-sided dice. Suppose each die is fair.
Intuitively, what is the probability that the two dice sum to 6? $\boxed{(D1 + D2 = 6)}$

Die 2 (RED)

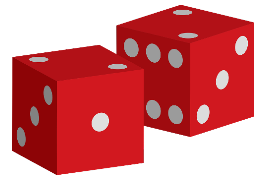
Die 1 (BLUE)

	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

each outcome has same probability

$Pr(\omega) = \frac{1}{16}$

EXAMPLE: WEIRD DICE (EVENTS)



THINK BACK TO THE 4-SIDED DICE. SUPPOSE EACH DIE IS FAIR.

INTUITIVELY, WHAT IS THE PROBABILITY THAT THE TWO DICE SUM TO 6? (D1 + D2 = 6)

DIE 2 (RED)

	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4) B
3	(3, 1)	(3, 2)	(3, 3) B	(3, 4)
4	(4, 1)	(4, 2) B	(4, 3)	(4, 4)

DIE 1 (BLUE)

EACH OF THE 16 OUTCOMES IS EQUALLY LIKELY.

3/16.

Equally Likely Outcomes

If Ω is such that outcomes are equally likely, then for any event $E \subseteq \Omega$,

$$P(E) = \frac{|E|}{|\Omega|} = \sum_{\omega \in E} \Pr(\omega) = \frac{|E|}{|\Omega|}$$

$$\Pr(\omega) = \frac{1}{|\Omega|}$$

$$\Pr(\omega) = \frac{1}{|\Omega|}$$

COIN TOSSING

$\Omega =$ all possible sequences of length 100, each elt H or T

$= \{H,T\}^{100}$

TOSS A COIN 100 TIMES. EACH OUTCOME IS EQUALLY LIKELY. WHAT IS THE PROBABILITY OF SEEING 50 HEADS?

$\text{Prob}(\omega) = \frac{1}{|\Omega|}$

$|\Omega| = 2^{100}$

(a) $\frac{1}{2}$

(b) $\frac{1}{2^{50}}$

(c) $\boxed{\frac{\binom{100}{50}}{2^{100}}}$

(d) I don't know.

$|E| = \binom{100}{50}$

$E: \{$sequences with exactly 50 Hs$\}$

$E:$ First coin toss is H

$\frac{2^{99}}{2^{100}} = \frac{1}{2}$

Non-equally likely outcomes

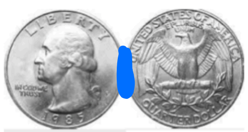
2 coin tosses

$\Omega = \{H, T\}$

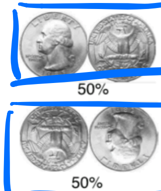
0.45   0.55

H: 45%

T: 55%



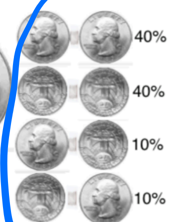
Glued coins



$\{HH, HT, TH, TT\}$

0   0.5   0.5   0

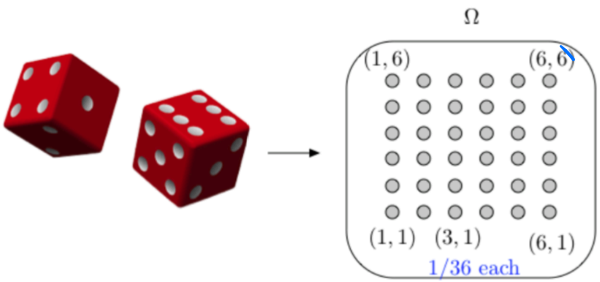
Attached coins



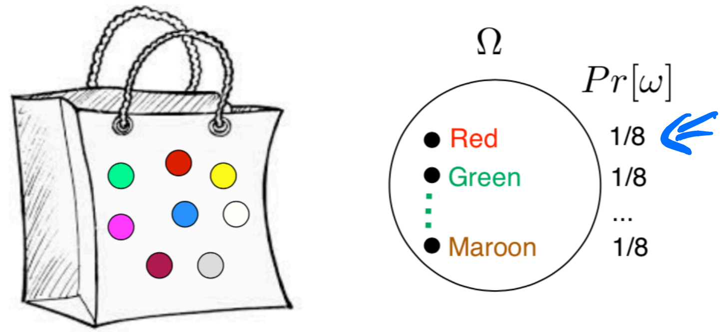
$P(\{HH, TT\}) = 0.4 + 0.4$

$= 0.8$

More examples – uniform probability spaces

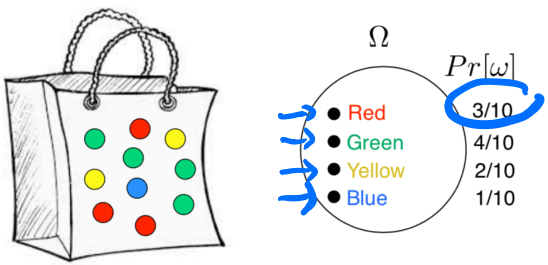


Physical Experiment   Probability Model

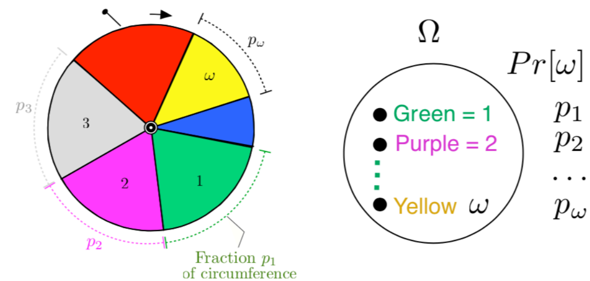


Physical experiment   Probability model

Nonuniform probability spaces



Physical experiment Probability model



Physical experiment Probability model

Birthday Paradox.

365 days in a year $\quad U = \{1, 2, \dots, 365\}$

drawing sample of size n from U with replacement

$$\begin{matrix} 1, & 2, & \dots & , n \\ \boxed{b_1}, & b_2, & & , b_n \end{matrix}$$

$\in U$

outcomes

$$\Omega = \{(b_1, b_2, \dots, b_n) \mid b_i \in U \ \forall i\}$$

$|\Omega| = ?$

- a) 365
- b) $P(365, n)$
- c) $\binom{365}{n}$
- d) $\underline{365^n}$ $\Leftarrow$

$$Pr(\omega) = \frac{1}{365^n}$$

Pr(at least 2 people with same birthday)

$= 1 -$ Pr(no 2 people have same bday)

$$\Pr\left(\text{no 2 people have same bday}\right) = \frac{|A|}{|\Omega|}$$

$$= \frac{365 \cdot 364 \cdots (365-n+1)}{365^n}$$

$|A| = ?$

a) 365

b) $P(365, n)$

c) $\binom{365}{n}$

d) 365^n

$\underline{\quad}_1 \ \underline{\quad}_2 \ \underline{\quad}_3 \ \cdots \ \underline{\quad}_n$

$\underline{M3}$	$\underline{A10}$	$\underline{M13}$
1	2	3

$\underline{A10}$	$\underline{M3}$	$\underline{M13}$
1	2	3

Anna	Tushar
Mar19	Nov4
Nov4	Mar19

$n = 23$ Pr(at least 2 same bday) ≥ 0.5

$n = 30$ ≥ 0.99

> 365 people in class

Pr(at least 2 with same bday) $= 1$

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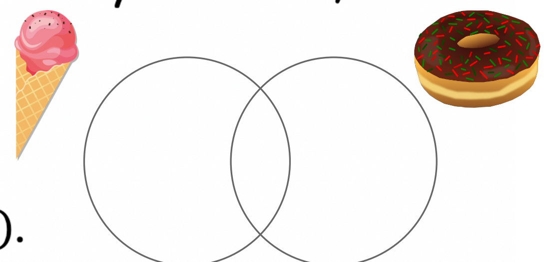

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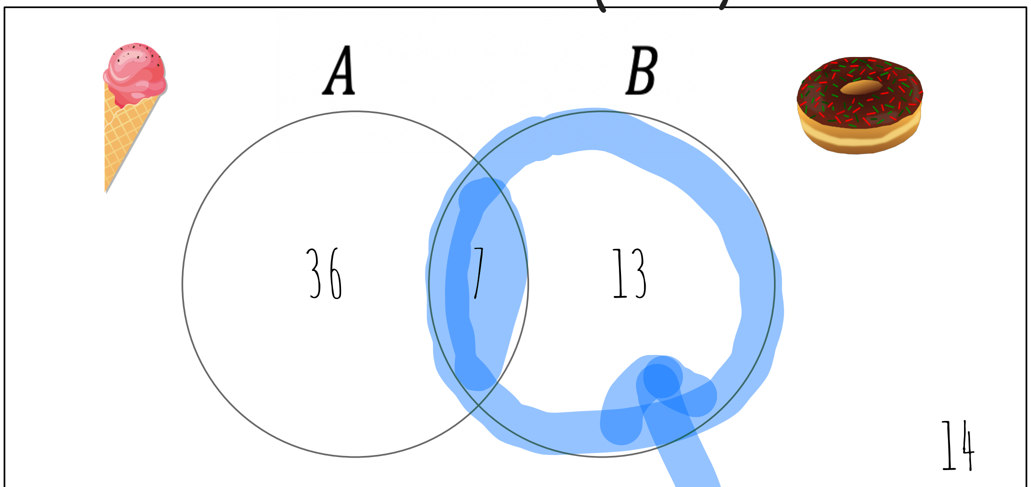
Conditional Probability

slides mostly by Alex Tsun

Conditional Probability (idea)



70 people

pick random person set

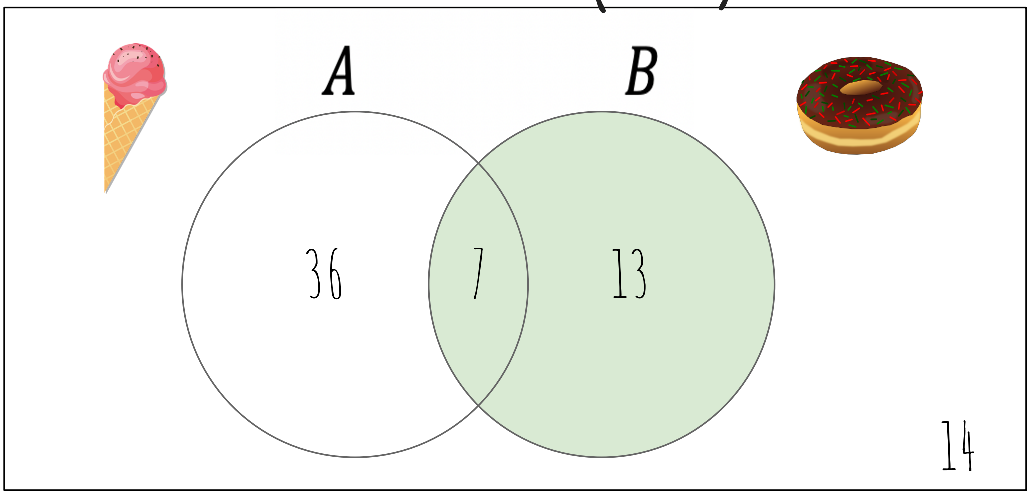
$Pr(w) = \frac{1}{70}$

What's the probability that someone likes ice cream **GIVEN** they like donuts?

$P(\text{likes IC}) = \frac{36+7}{70}$

$\frac{7}{20}$

Conditional Probability (idea)



What's the probability that someone likes ice cream **given** they like donuts?

$$P(A|B) = \frac{7}{20} = \frac{|A \cap B|}{|B|} = \frac{|A \cap B|/|\Omega|}{|B|/|\Omega|} = \frac{P(A \cap B)}{P(B)}$$

Conditional Probability

Conditional Probability: The (conditional) probability of A given an event B happened is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

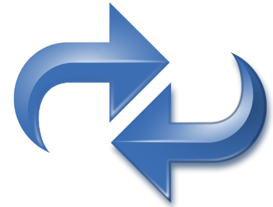
assume $Pr(B) \neq 0$

An equivalent and useful formula is $P(A \cap B) = P(A|B)P(B)$.

Conditional Probability (Reversal)

Does $P(A|B) = P(B|A)$?

Conditional Probability (intuition)



Does $P(A|B) = P(B|A)$? No!!

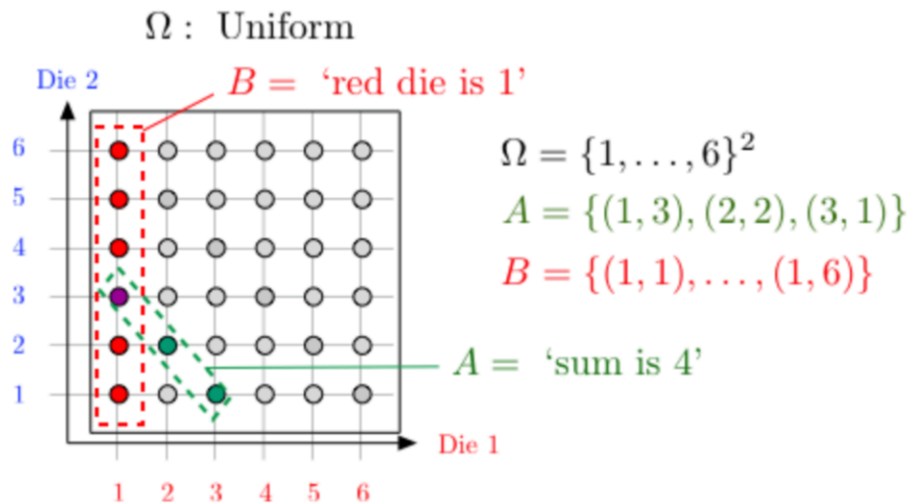
Let A be the event you are wet.
Let B be the event you are swimming.

$$P(A|B) = 1$$

$$P(B|A) \neq 1$$

Fun with conditional probability

- Toss a red die and a blue die. All outcomes equally likely. What is Pr(B | A)? What is Pr(B)?



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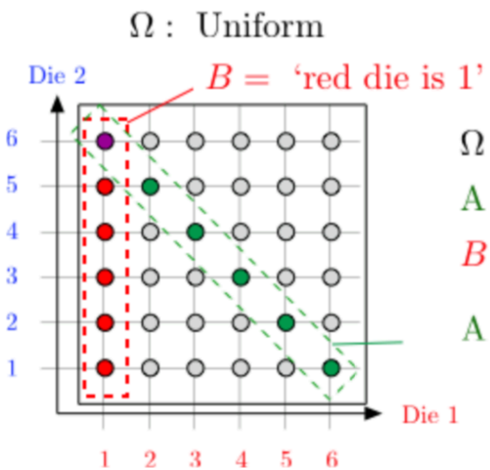

$\Omega = \{1, \ldots, 6\}^2$

$A = \{(1, 6), \ldots, (6,1)\}$

$B = \{(1, 1), \ldots, (1, 6)\}$

A = 'sum is 7'

Pr(B|A)

(a) 1/7

(b) 1/6

(c) 1/36

(d) $\frac{11}{36}$

37

Gambler's fallacy

- Flip a fair coin 51 times. All outcomes equally likely.
- A = "first 50 flips are heads"
- B = "the 51st flip is heads"

- Pr (B | A) = ?