Announcements
- pset 2 posted after lecture
- partner finding form due by noon.

2. Probability
Intro to Discrete Probability

安娜·卡尔林

与许多来自亚历克斯·田和CS70在伯克利的幻灯片
AGENDA

- Definitions
- Axioms
- Equally Likely Outcomes
- Beyond equally likely outcomes
- Conditional Probability
**Definitions**

Sample Space: The set $\Omega$ of all possible outcomes of an experiment.

- Single coin flip: $\Omega = \{H, T\}$
- Two coin flips: $\Omega = \{HH, HT, TH, TT\}$
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$

Probability space assigns a number $P(\omega)$ to each $\omega \in \Omega$. 
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**Event:** Any subset $E \subseteq \Omega$.

- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
- Rolling an even number: $E = \{2, 4, 6\}$
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Mutually Exclusive: Events $E$ and $F$ are mutually exclusive if $E \cap F = \emptyset$ (i.e., they can’t simultaneously happen).

- $E = \{2, 4, 6\}$ and $F = \{1, 3\}$, then $E \cap F = \emptyset$. 
**Example: weird dice (Sample Space)**

Suppose I roll two 4-sided dice. Here is the sample space (set of possible outcomes):

<table>
<thead>
<tr>
<th>Die 2 (Red)</th>
<th>1</th>
<th>2</th>
<th>3</th>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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</tbody>
</table>
Example: weird dice (Events)

Let $D_1$ be the value of the blue die, and $D_2$ the value of the red die. What outcomes match these events?

A. $D_1 = 1$
B. $D_1 + D_2 = 6$
C. $D_1 = 2 \times D_2$
### Example: Weird Dice (Events)

**Are A and B mutually exclusive?**

**Are B and C mutually exclusive?**

**A.** \( D_1 = 1 \)

**B.** \( D_1 + D_2 = 6 \)

**C.** \( D_1 = 2 * D_2 \)

#### Die 1 (Blue)

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<thead>
<tr>
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<tbody>
<tr>
<td>1</td>
<td>(1, 1) (^A)</td>
<td>(1, 2) (^A)</td>
<td>(1, 3) (^A)</td>
<td>(1, 4) (^A)</td>
</tr>
<tr>
<td>2</td>
<td>(2, 1) (^C)</td>
<td>(2, 2)</td>
<td>(2, 3)</td>
<td>(2, 4) (^B)</td>
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<tr>
<td>3</td>
<td>(3, 1)</td>
<td>(3, 2)</td>
<td>(3, 3) (^B)</td>
<td>(3, 4)</td>
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<tr>
<td>4</td>
<td>(4, 1)</td>
<td>(4, 2) (^B) (^C)</td>
<td>(4, 3)</td>
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</table>

#### Die 2 (Red)
**Example: Weird Dice (Events)**

**Are A and B mutually exclusive?**

A. \( D_1 = 1 \)

B. \( D_1 + D_2 = 6 \)

C. \( D_1 = 2 \cdot D_2 \)

**Are B and C mutually exclusive?**

- (a) Yes
- (b) Yes
- (c) No

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<td>( (1, 4) ) (A)</td>
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<td>2</td>
<td>( (2, 1) ) (C)</td>
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**Example: Weird Dice (Mutually Exclusive)**

Are A and B mutually exclusive?

Yes. $A \cap B = \emptyset$ (No overlap)

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**Example: weird dice (mutually exclusive)**

Are **B** and **C** mutually exclusive?

NO. **B** and **C** could happen at the same time (4, 2)

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Die 1 (blue)

Die 2 (red)
Random Picture
Let $\Omega$ denote the sample space and $E, F \subseteq \Omega$ be events.

Axiom 1 (Nonnegativity): $P(E) \geq 0$.
Axiom 2 (Normalization): $P(\Omega) = 1$.
Axiom 3 (Countable Additivity): If $E$ and $F$ are mutually exclusive, then $P(E \cup F) = P(E) + P(F)$.

$$\Pr(E) = \sum_{\omega \in E} \Pr(\omega)$$

$$(1,2)$$
Axioms of Probability & Their Consequences

Let $\Omega$ denote the sample space and $E, F \subseteq \Omega$ be events.

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Corollary 1 (Complementation): $P(E^c) = 1 - P(E)$.

$\Omega = E \cup E^c$

$1 = \Pr(\Omega) = \Pr(E) + \Pr(E^c)$
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Corollary 2 (Monotonicity): If $E \subseteq F$, $P(E) \leq P(F)$.

$$P(F) = P(E) + P(F \mid E) \geq 0$$

$\Rightarrow P(F) \geq P(E)$
Axioms of Probability & Their Consequences

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**Corollary 3 (Inclusion-Exclusion)**: $P(E \cup F) = P(E) + P(F) - P(E \cap F)$. 
**Example: Weird Dice (Events)**

Think back to the 4-sided dice. Suppose each die is fair. Intuitively, what is the probability that the two dice sum to 6? \((D_1 + D_2 = 6)\)

Each outcome has same probability

\[
\Pr(w) = \frac{1}{16}
\]

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Example: weird dice (Events)

Think back to the 4-sided dice. Suppose each die is fair. Intuitively, what is the probability that the two dice sum to 6? \((D_1 + D_2 = 6)\)

Each of the 16 outcomes is equally likely.

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3/16.
Equally Likely Outcomes

If $\Omega$ is such that outcomes are equally likely, then for any event $E \subseteq \Omega$,

$$P(E) = \frac{|E|}{|\Omega|} = \sum_{\omega \in E} \Pr(\omega) = \frac{|E|}{|\Omega|}$$
Coin tossing

Toss a coin 100 times. Each outcome is equally likely. What is the probability of seeing 50 heads?

\[ S = \text{all possible sequences of length 100} \]
\[ \text{each elt } H = \frac{1}{2} \]
\[ = \{H,T\}^{100} \]

\[ \text{Prob}(w) = \frac{1}{100!} \]

\[ |S| = 2^{100} \]
\[ |E| = \binom{100}{50} \]

E: first coin toss is H
\[ \frac{2^{99}}{2^{100}} = \frac{1}{2} \]

(a) \( \frac{1}{2} \)
(b) \( \frac{1}{250} \)
(c) \( \frac{\binom{100}{50}}{2^{100}} \)
(d) I don't know.
Non-equally likely outcomes

\( N = \{ H, T \} \)

- \( H: 45\% \)
- \( T: 55\% \)

2 coin tosses

\[ P(HH, HT, TH, TT) = 0.4 + 0.4 = 0.8 \]
More examples – Uniform probability spaces

Physical Experiment

Probability Model

Physical experiment

Probability model
Nonuniform probability spaces

Physical experiment

Probability model

\[ \Omega \]

- Red: 3/10
- Green: 4/10
- Yellow: 2/10
- Blue: 1/10

\[ Pr[\omega] \]

Physical experiment

Probability model

\[ \Omega \]

- Green = 1
- Purple = 2
- Yellow

\[ Pr[\omega] \]

\[ p_1 \]
\[ p_2 \]
\[ p_\omega \]
Birthday Paradox.

365 days in a year \( U = \{1, 2, \ldots, 365\} \)

drawing sample of size \( n \) from \( U \) with replacement.

\[
\begin{bmatrix}
1, 2, \ldots, n \\
b_1, b_2, \ldots, b_n
\end{bmatrix}
\]

c \in U \rightarrow \text{outcomes}

\[ S = \{(b_1, b_2, \ldots, b_n) \mid b_i \in U\} \]

\( \left| S \right| = ? \)

\[
\text{Pr}(w) = \frac{1}{365^n}
\]

\[
\text{Pr}(\text{at least 2 people with same birthday}) = 1 - \text{Pr}(\text{no 2 people have same bday})
\]
\[ \Pr(\text{no 2 people have the same bday}) = \frac{|A|}{121} \]

\[ = \frac{365 \cdot 364 \cdots (365-n+1)}{365^n} \]

\[ |A| = 2^7 \]

- (a) \(365\)
- (b) \(P(365,n)\)
- (c) \(\binom{365}{n}\)
- (d) \(365^n\)

- Anna Tushar
- Mar 19 Nov 4
- Nov 4 Mar 19

\[ \frac{M_3}{1} \quad \frac{A_{10}}{2} \quad \frac{M_{13}}{3} \]
n = 23 \quad \Pr(\text{at least } 2 \text{ same bday}) \geq 0.5

n = 30 \quad \geq 0.99

> 365 \text{ people in class}

\Pr(\text{at least } 2 \text{ with same bday}) = 1
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Conditional Probability

slides mostly by Alex Tsun
Conditional Probability (idea)

What's the probability that someone likes ice cream given they like donuts?

\[ P(\text{likes IC} | B) = \frac{36 + 7}{70} \]

70 people pick random person set
\[ Pr(I) = \frac{1}{70} \]

\[ \frac{7}{20} \]
Conditional Probability (idea)

What's the probability that someone likes ice cream given they like donuts?

\[ P(A|B) = \frac{\frac{|A \cap B|}{|B|}}{\frac{|B|}{|\Omega|}} = \frac{\frac{|A \cap B|}{|\Omega|}}{P(B)} \]
**Conditional Probability**

**Conditional Probability**: The (conditional) probability of $A$ given an event $B$ happened is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{assume } P(B) \neq 0$$

An equivalent and useful formula is $P(A \cap B) = P(A|B)P(B)$. 
Conditional Probability (Reversal)

Does $P(A|B) = P(B|A)$?
Conditional Probability (intuition)

Does $P(A|B) = P(B|A)$? **No!!**

Let $A$ be the event you are wet.
Let $B$ be the event you are swimming.

$P(A|B) = 1$

$P(B|A) \neq 1$
Fun with conditional probability

- Toss a red die and a blue die. All outcomes equally likely. What is $\Pr(B \mid A)$? What is $\Pr(B)$?
Fun with conditional probability

- Toss a red die and a blue die. All outcomes equally likely. What is $\Pr(B \mid A)$?

$\Omega$: Uniform

$B = \text{‘red die is 1’}$

$\Omega = \{1, \ldots, 6\}^2$

$A = \{(1, 6), \ldots, (6,1)\}$

$B = \{(1,1), \ldots, (1,6)\}$

$A = \text{‘sum is 7’}$

(a) $\frac{1}{7}$

(b) $\frac{1}{6}$

(c) $\frac{1}{36}$
Gambler’s fallacy

- Flip a fair coin 51 times. All outcomes equally likely.
- A = “first 50 flips are heads”
- B = “the 51st flip is heads”

Pr (B | A) = ?