Announcements

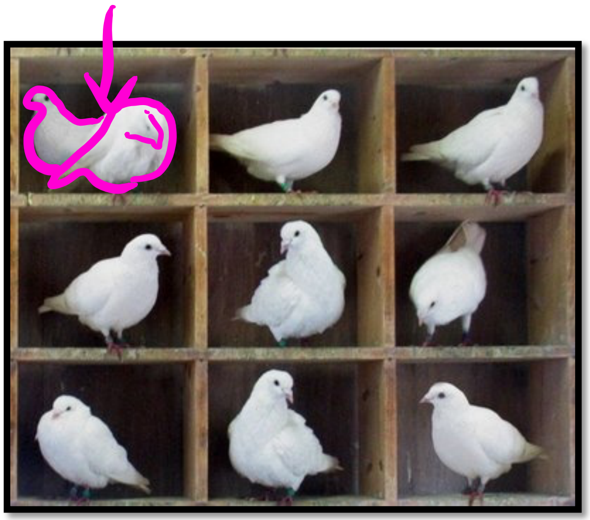
- homework solutions - use section solutions as model
- feedback.
- partner-finding form.

More counting

Anna Karlin

Most of the Slides created by Alex Tsun

Pigeonhole principle

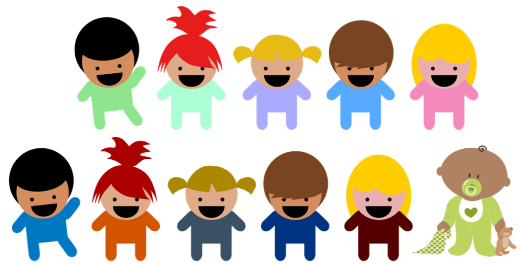


9 holes
10 pigeons.

Pigeonhole Principle: idea

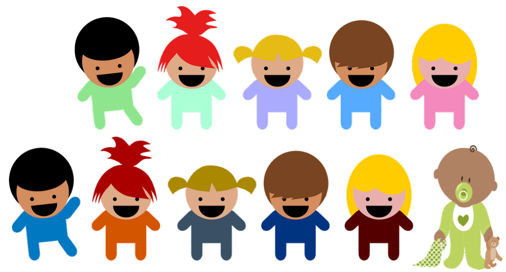
$\frac{11}{3} = 3\frac{2}{3}$

round up.



Suppose we split 11 children up into 3 groups and each group gets a cake to share. What is the largest number of children that will need to share a cake?

Pigeonhole Principle: idea



If 11 children have to share 3 cakes, at least one cake must by at least how many children? 4 (11/3 but rounded up)

Pigeonhole principle (PHP)

If there are n pigeons we want to put into k pigeonholes (where $n > k$), then at least one pigeonhole must contain at least 2 pigeons.

More generally, if there are n pigeons we want to put into k pigeonholes, then at least one pigeonhole must contain at least $\lceil n/k \rceil$ pigeons.

$\frac{n}{k}$ rounded up to next integer

The floor and ceiling functions

The floor function $\lfloor x \rfloor$ returns the largest integer $\leq x$ (i.e., rounds down).

$$\lfloor 2.5 \rfloor = 2 \qquad \lfloor 16.99999 \rfloor = 16 \qquad \lfloor 5 \rfloor = 5$$

The ceiling function $\lceil x \rceil$ returns the smallest integer $\geq x$ (i.e., rounds up).

$$\lceil 2.5 \rceil = 3 \qquad \lceil 9.000301 \rceil = 10 \qquad \lceil 5 \rceil = 5$$

Pigeonhole principle (PHP)

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More generally, if there are n pigeons we want to put into k pigeonholes, then at least one pigeonhole must contain at least $\lceil n/k \rceil$ pigeons.

$7, \ 10523, \ -54, \ 10^{10} \ \ldots.$

integers. 3

Use the PHP to show that in every set of 100 numbers, there are two whose difference is a multiple of 37.

Pigeonhole principle (PHP)

Use the PHP to show that in every set of 100 numbers, there are two whose difference is a multiple of 37.

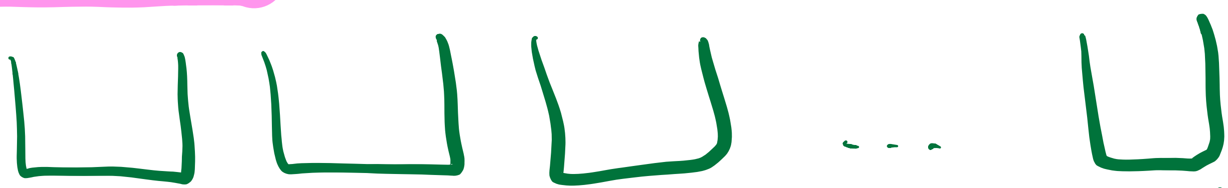
100 numbers $\Rightarrow$ 37 bins.

$2 < \boxed{\frac{100}{37}} < 3$

$i \longrightarrow i \bmod 37$

When solving a PHP problem:

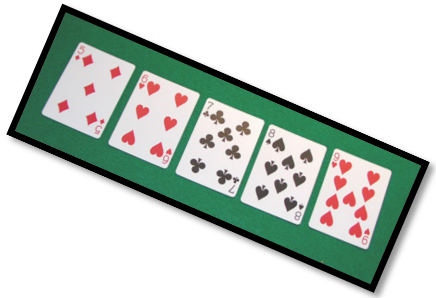
- Identify the pigeons
- Identify the pigeonholes
- Specify how pigeons are assigned to holes
- Apply the principle



0 mod 37 1 mod 37 2 mod 37 36 mod 37

Let's practice some more

Quick review of cards



52 total cards

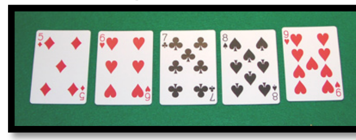
- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

Counting cards

- How many possible 5 card hands?

$$\binom{52}{5}$$

- A "straight" is five consecutive rank cards of any suit. How many possible straights?



1 lowest rank

2 suit lowest

3 suit 2nd lowest

4

5

6 suit for highest

$$10 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 10 \cdot 4^5$$

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

Counting cards

- How many possible 5 card hands?

$$\binom{52}{5}$$

- A flush is five card hand all of the same suit. How many possible flushes?

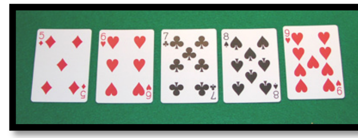
1 suit

2 5 ranks in that suit

$$4 \cdot \binom{13}{5}$$

Counting cards

A "straight" is five consecutive rank cards of any suit. How many possible straights?



$$10 \cdot 4^5 = 10,240$$

A flush is five card hand all of the same suit. How many possible flushes?

$$4 \cdot \binom{13}{5} = 5,148$$

How many flushes are not straights?

\# flushes − \# straight flushes

$4 \cdot \binom{13}{5} -$ lowest rank · suit

$10 \cdot 4$

(a) $11 \cdot \binom{12}{5} \cdot 4$

(b) $10 \cdot 4^5 - 4 \cdot 10$

(c) $4 \cdot \binom{13}{5} - 4 \cdot 10$

(d) I don't know

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

THE SLEUTH'S CRITERION (RUDICH)

FOR EACH OBJECT CONSTRUCTED, IT SHOULD BE POSSIBLE TO RECONSTRUCT THE UNIQUE SEQUENCE OF CHOICES THAT LED TO IT.

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

$\Rightarrow$ AC, AS, AS, AC, 5S

$$\binom{4}{3} \cdot \binom{49}{2}$$

First choose 3 Aces, then choose remaining two cards.

(a) correct
(b) overcounts
(c) undercounts

$\Rightarrow$ AC, AD, AS, AH, 2H

$\Rightarrow$ AC, AD, AS

$\Rightarrow$ AC, AS, AH

how many hands are overcounted

$\binom{48}{1}$ (non ace)

$\Rightarrow$ AD, AS, AH

$\Rightarrow$ AC, AD, AH

The Sleuth's criterion (Rudich)

For each object constructed, it should be possible to reconstruct the unique sequence of choices that led to it.

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

$\binom{4}{3} \cdot ($ … rst choose 3 Aces, then choose … ining two cards.

The Sleuth's criterion (Rudich)

For each object constructed, it should be possible to reconstruct the unique sequence of choices that led to it.

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

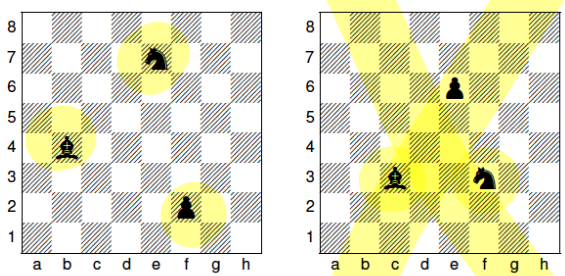
When in doubt, break set up into disjoint sets you know how to count and then use the sum rule.

\# hands with exactly 3 Aces + # hands with 4 aces

$$\binom{4}{3}\binom{48}{2} + \binom{4}{4}\cdot\binom{48}{1}$$

8 BY 8 CHESSBOARD

HOW MANY WAYS TO PLACE A PAWN, A BISHOP AND A KNIGHT SO THAT NONE ARE IN THE SAME ROW OR COLUMN



(a) valid (b) invalid

(a) $\binom{64}{3}$

(b) $8^2 7^2 6^2$

(c) $\binom{8}{3}\binom{8}{3}$

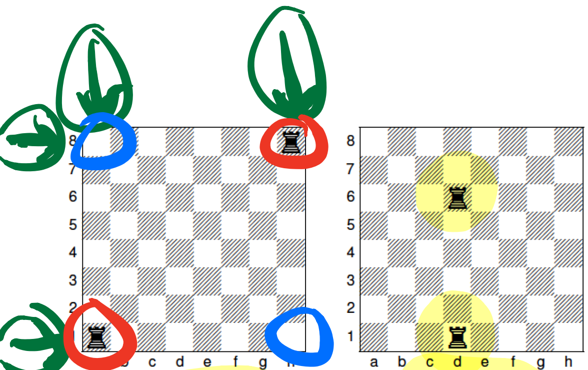
(d) I don't know

place pawn 8^2 · place bishop 7^2 · place knight 6^2

Rooks on chessboard

indistinguishable

How many ways to place two identical rooks on a chessboard so that they don't share a row or a column



(a) valid (b) invalid

(a) $8^2 \cdot 7^2$ overcounts

(b) $\binom{8}{2}\binom{8}{2}$ undercounts

(c) $\boxed{\frac{8^2 \cdot 7^2}{2}}$

(d) I don't know

choose 2 rows choose 2 cols

(8 (8

(2) (2)

DOUGHNUTS

You go to Top Pot to buy a dozen donuts. Your choices are

Chocolate, Lemon-filled, Maple, Glazed, plain

How many ways are there to choose a dozen doughnuts when doughnuts of the same type are indistinguishable?

$\star\ \star\ \star \mid \star\ \star\ \star \mid\mid \star\ \star \mid \star\ \star\ \star$

choc L M G P

$\binom{16}{4}$

$$\binom{12+4}{4}$$

Stars and Bars/Divider method

The number of ways to distribute n indistinguishable balls into k distinguishable bins is

$$\binom{n+(k-1)}{k-1} = \binom{n+(k-1)}{n}$$

DOUGHNUTS

You go to top pot to buy a dozen donuts. Your choices are

Chocolate, Lemon-filled, sugar, Glazed, plain

How many ways are there to choose a dozen doughnuts when you want at least 1 of each type?

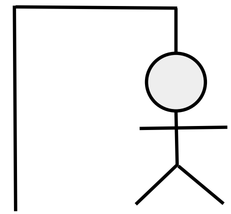
7

$$\boxed{\binom{7+4}{4}} \quad \binom{11}{4}$$

$\star \mid \star \ \star \mid \mid \star \ \star \ \star \mid \star$

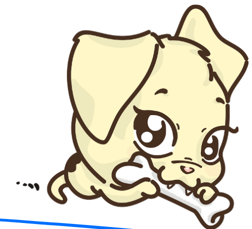
$\boxed{C, L, S, G, P}$

2 3 1 4 2



ANAGRAMS

How many ways can you arrange the letters in "Godoggy"?



$N = 7$ letters, $K = 4$ types $\{G, O, D, Y\}$

$N_1 = 3,\ N_2 = 2,\ N_3 = 1,\ N_4 = 1$

$$\frac{7!}{3!\,2!\,1!\,1!} = \binom{7}{3,2,1,1}$$

Multinomial Coefficients

If we have k types of objects (n total), with n_1 of the first type, n_2 of the second, ..., and n_k of the k^{th}, then the number of arrangements possible is

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!\, n_2!\, \ldots\, n_k!}$$

COMBINATORIAL ARGUMENT/PROOF

- LET S BE A SET OF OBJECTS
- SHOW HOW TO COUNT |S| ONE WAY => |S|=N
- SHOW HOW TO COUNT |S| ANOTHER WAY =>. |S| = M

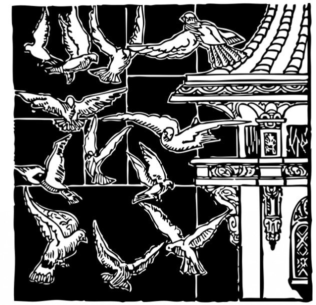
CONCLUDE N=M

$$\binom{n}{r} = \binom{n}{n-r}$$

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

$$\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$$

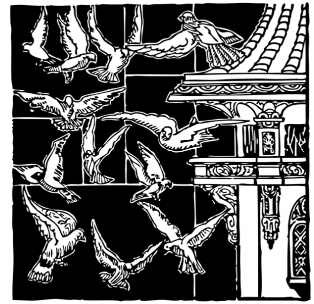
Combinatorial Proofs: example



Show that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Consider the set of numbers $\{1,2,\ldots,n\}$.

Combinatorial Proofs: Example



Show that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

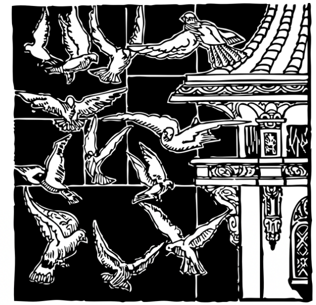
Consider the set of numbers $\{1, 2, \ldots, n\}$.

Left Side: Counts the number of subsets of size k.

Right Side: Two cases. We either include the number 1 or not.

- If we include the number 1, we need to choose $k-1$ out of the remaining $n-1$.
- If we don't include it, we need to choose k out of the remaining $n-1$.

Combinatorial Proofs: example



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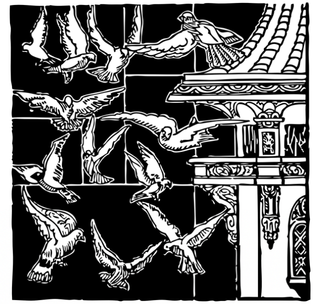

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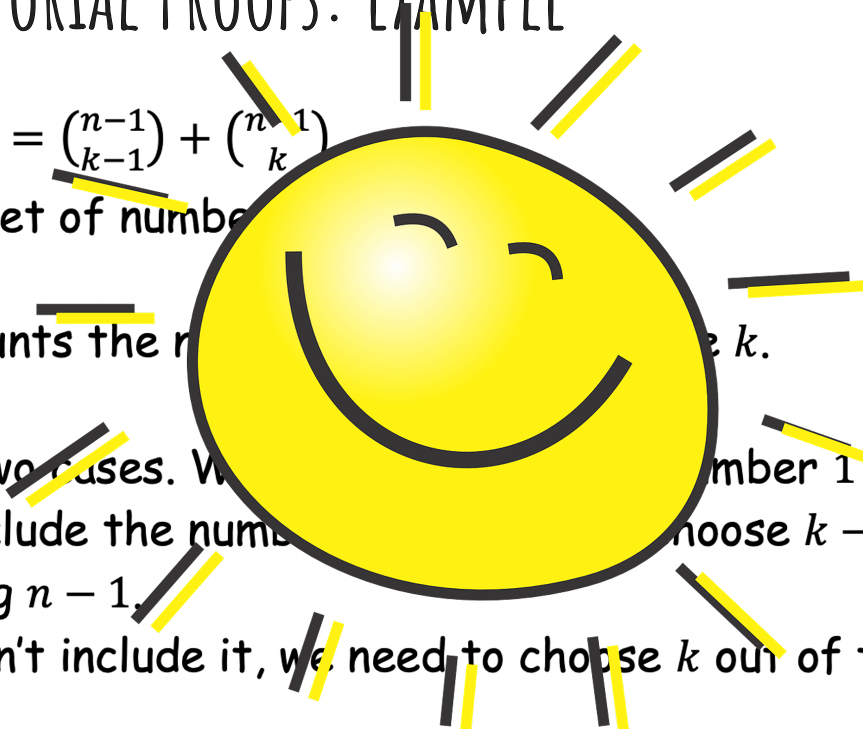

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Right Side: Two cases. W … mber 1 or not.

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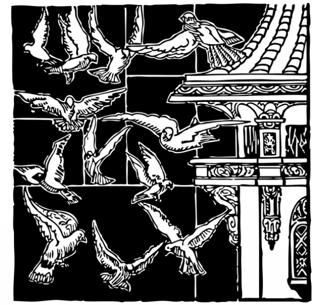
THE ALTERNATIVE....

Show that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.



$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-1-k)!}$$

$$= 20\ years\ later\ \ldots$$

$$= \frac{n!}{k!\,(n-k)!}$$

$$= \binom{n}{k}$$

Tools and concepts

- Sum rule, Product rule
- Permutations, combinations
- Inclusion-exclusion
- Binomial Theorem
- Combinatorial proofs
- Pigeonhole principle
- Stars and bars

Counting is NOT for kindergarteners