Announcements
1. Social hour: today at 5:30pm!
2. Textbook
3. Theorems & Definitions
4. office hours

Pace?
(a) too slow
(b) about right
(c) a bit too fast
(d) way too fast

More Counting

$$\frac{7!}{3!\,2!\,1!\,1!}$$

cards → 3 1 2

2 → 3 2 1

$$\binom{7}{3}\binom{4}{2}\cdots$$

Anna Karlin

Most of the Slides created by Alex Tsun

Product rule

If S is a set of sequences of length k for which there are

- n_1 choices for the first element of the sequence
- n_2 choices for the second element of the sequence given any particular choice for the first,
- n_3 choices for the 3rd given any particular choice for 1st and 2nd
- ….

Then $|S| = n_1 \times n_2 \times \ldots \times n_k$

Permutations

The number of orderings of n **distinct** objects is

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$$

Read as "n factorial"

K-Permutations

If we want to arrange **only** k out of n distinct objects, the number of ways to do so is

$$P(n,k) = n \times (n-1) \times (n-2) \times \ldots \times (n-k+1) = \frac{n!}{(n-k)!}$$

Read as "n pick k"

Combinations/Binomial Coefficients

If we want to select k out of n distinct objects, where order does not matter, the number of ways to do so is

$$C(n,k) = \binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}$$

Both Acceptable

Read as "n choose k"

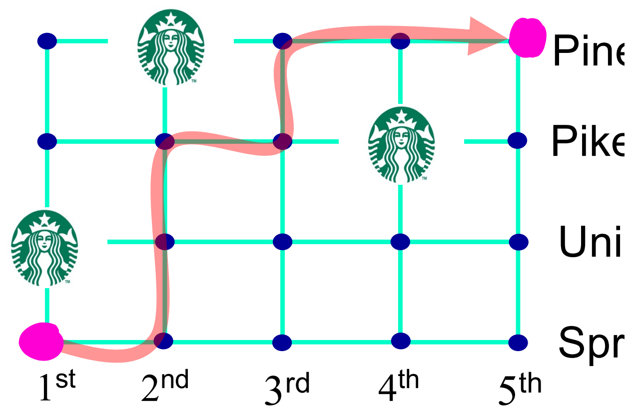
How many ways to walk from 1st and Spring to 5th and Pine?

Only going north and east

7 steps

4 E

3 N

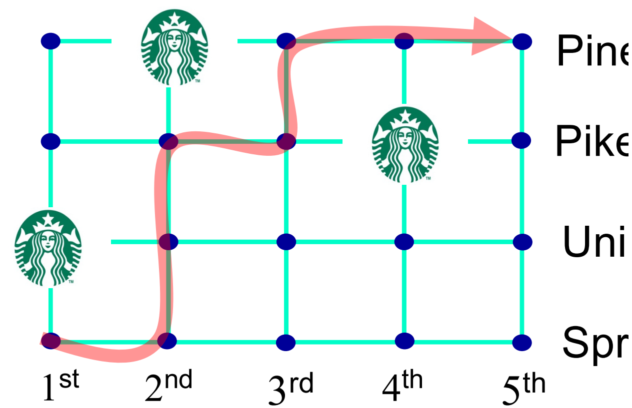


ENNENEE

_ _ _ _ _ _ _

How many ways to walk from 1st and Spring to 5th and Pine?

Only going north and east



(a) 2^7

(b) $\binom{7}{3}$

(c) $\binom{7}{4}$

(d) $P(7,3)$

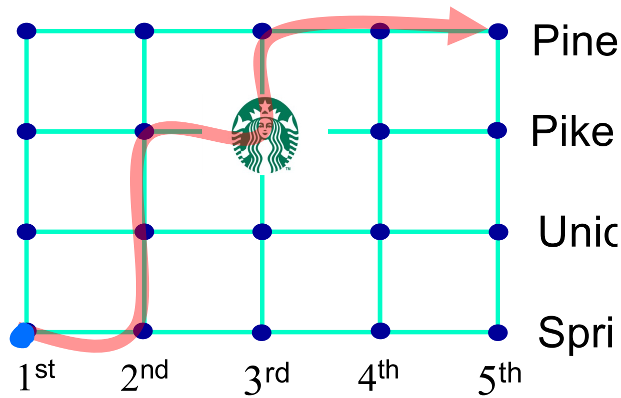
choosing 3 of the 7 steps in which to go N

$2,5,6 \Rightarrow N$

$5,6,2$

How many ways to walk from 1^{st} and Spring to 5^{th} and Pine, stopping at the Starbucks on 3^{rd} and Pike

Only going north and east



(a) $\binom{7}{3}$

(b) $\binom{7}{3}\binom{7}{1}$

(c) $\binom{4}{2}\cdot\binom{3}{1}$

(d) $\binom{4}{2}\cdot\binom{3}{2}$

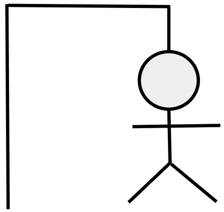
1st & S → SB — 4 steps, 2 N

SB → 5th Pine — 3 steps, 2 E

$\binom{4}{2} \cdot 3$

Random Picture

ANAGRAMS



How many ways can you arrange the letters in "MATH"?

(a) 4^4

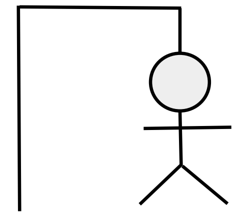
(b) $4!$

(c) $\binom{26}{4}$

MATH

$\underline{4} \cdot \underline{3} \quad \underline{2} \quad \underline{1}$

ANAGRAMS

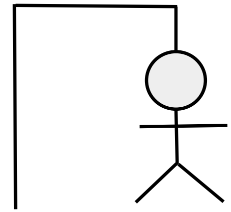


HOW MANY WAYS CAN YOU ARRANGE THE LETTERS IN "MATH"?

4! = 24 SINCE THEY ARE DISTINCT OBJECTS!

MATH

ANAGRAMS



How many ways can you arrange the letters in "MUUMUU"?

U	M	U	U	M	U

(a) $6!$

(b) $\binom{6}{2}\cdot\binom{4}{4}$

(c) $\binom{6}{4}$

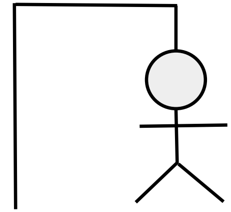
(d) $6\cdot 5$

choose positions M's
then U's

choose positn for U's
then M's

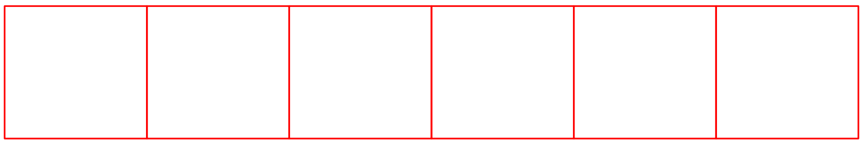
$\binom{4}{4} = \frac{4!}{4!\,0!}$

$0! \stackrel{?}{=} 1$



ANAGRAMS

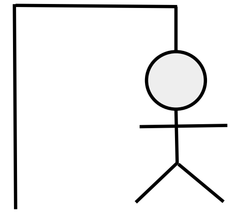
How many ways can you arrange the letters in "MUUMUU"?



Choose where the 2 M's go, and then the U's are set. Or choose where the 4 U's go, and then the M's are set.

Either way, we get $\binom{6}{2} \cdot \binom{4}{4} = \binom{6}{4} \cdot \binom{2}{2} = \frac{6!}{2!4!}$

Another way to think about it



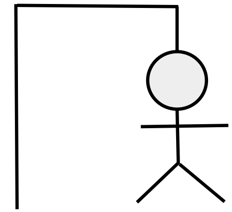
How many ways can you arrange the letters in "MUUMUU"?

1 2

$M_1 U_1 U_2 M_2 U_3 U_4$

$$\frac{6!}{2!\,4!} = \binom{6}{4} = \binom{6}{2} = \binom{6}{4}\binom{2}{2}$$

Anagrams



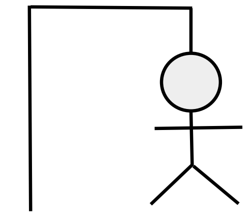
How many ways can you arrange the letters in "Muumuu"?

$$\binom{6}{2}\cdot\binom{4}{4}=\binom{6}{4}\cdot\binom{2}{2}=\frac{6!}{2!4!}$$

Another interpretation:

Arrange the 6 letters as if they were distinct. Then divide by 4! and 2! to account for 4 duplicate O's and 2 duplicate P's.

ANAGRAMS



How many ways can you arrange the letters in "Godoggy"?

G	O	D	Y
3	2	1	1

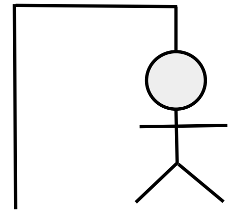
$G_1 G_2 G_3$

$O_1 O_2$

D

Y

ANAGRAMS



How many ways can you arrange the letters in "Godoggy"?

(a) $7!$

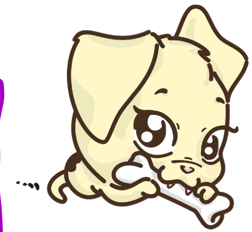
(b) $\frac{7!}{3!}$

(c) $\frac{7!}{3!\,2!\,1!\,1!}$

(d) $\binom{7}{3}\binom{5}{2}3!$

G	O	D	Y
3	2	1	1

$G_1\,G_2\,G_3$
$O_1\,O_2$
D
Y



$$\frac{7!}{3!\,2!\,1!\,1!}$$

_ O G _ G G O

$$\binom{7}{3}\binom{4}{2}2!$$

Final set of concepts...

- Binomial Theorem
- Inclusion-Exclusion
- Stars and Bars/Divider Method
- Pigeonhole principle

$(x+y)^{10}$ $(x+y)^n$ integer n

BINOMIAL THEOREM: IDEA

$n=2$

$$(x+y)^2 = (x+y)(x+y)$$

$$xx + xy + yx + yy$$

$$x^2 + 2xy + y^2$$

$$\binom{2}{0}x^2 + \binom{2}{1}xy + \binom{2}{2}y^2$$

1 2 1

Binomial theorem: idea

$$(x+y)^4 = (x+y)(x+y)(x+y)(x+y)$$

$$xxxx + yyyy + xyxy + yxyy + \cdots$$

x^2y^2 $\quad$ xy^3

(a) 4

(b) $\binom{4}{1}$

(c) $\binom{4}{3}$

(d) 3

x^4 $\quad$ x^3y $\quad$ x^2y^2 $\quad$ xy^3 $\quad$ y^4

$$\binom{4}{1} = \binom{4}{3} = 4$$

Binomial theorem: idea

$$(x+y)^4 = (x+y)(x+y)(x+y)(x+y)$$

$$xxxx + yyyy + xyxy + yxyy + \cdots$$

$$(x+y)^n = \underset{1st}{(x+y)}\underset{2nd}{(x+y)}\cdots\underset{n^{th}}{(x+y)}$$

Binomial theorem: idea

$$(x+y)^4 = (x+y)(x+y)(x+y)(x+y)$$

Each term is of the form $x^k y^{n-k}$ (in our case, $n = 4$), since we multiply exactly n variables, either x or y.

How many times do we get $x^k y^{n-k}$? The number of ways to choose k of them to produce x (the rest will be y).

$$\binom{n}{k} = \binom{n}{n-k}$$

Binomial Theorem

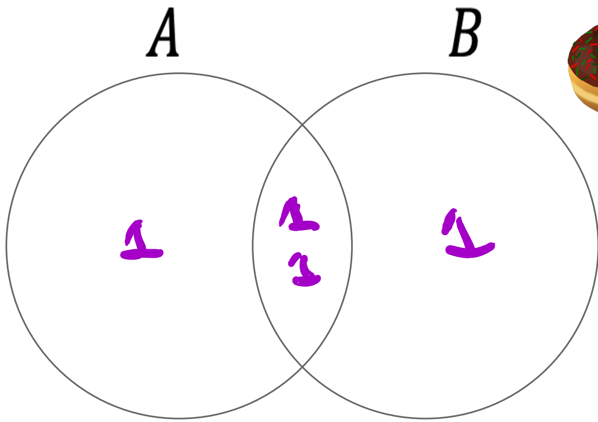
Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$k=0$ y^n

$k=1$ $x\,y^{n-1}$

Inclusion-Exclusion: Idea



$|A| = 43$

$|B| = 20$

$|A \cap B| = 7$

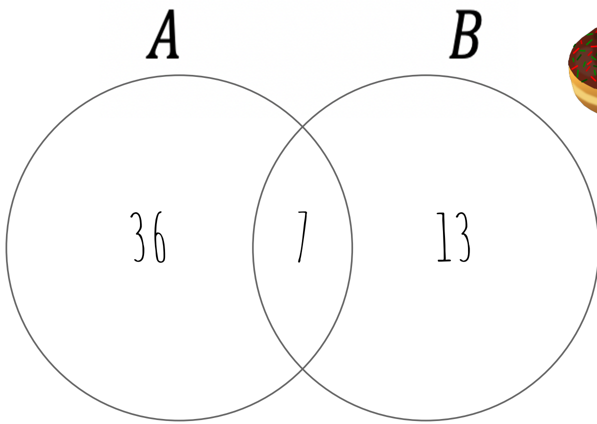
$|A \cup B| = ???$

$43 + 20 - 7$

$A \cup B$: set of people that like IC or D or both

$|A \cup B| = |A| + |B| - |A \cap B|$

Inclusion-Exclusion: idea

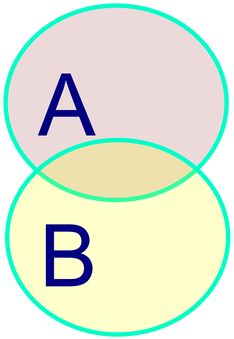


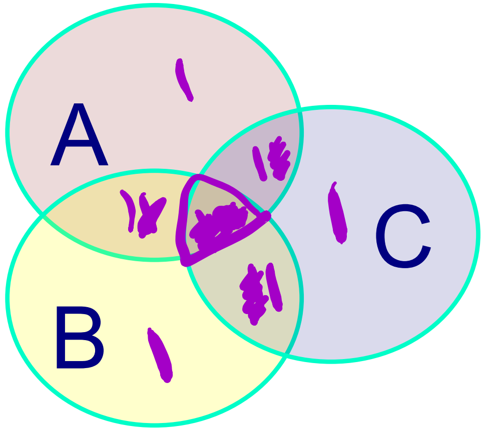
$$|A| = 43$$
$$|B| = 20$$
$$|A \cap B| = 7$$
$$|A \cup B| = ???$$

How many people like ice cream or donuts?

$$|A \cup B| = 36 + 7 + 13 = 56 = 43 + 20 - 7 = |A| + |B| - |A \cap B|$$

Inclusion-Exclusion





$|A \cup B| = |A| + |B| - |A \cap B|$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Inclusion-Exclusion

Let A, B be sets. Then,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

In general, if $A_1, A_2, \ldots, A_n$ are sets, then

$$|A_1 \cup \ldots \cup A_n| = singles - doubles + triples - quads + \cdots$$

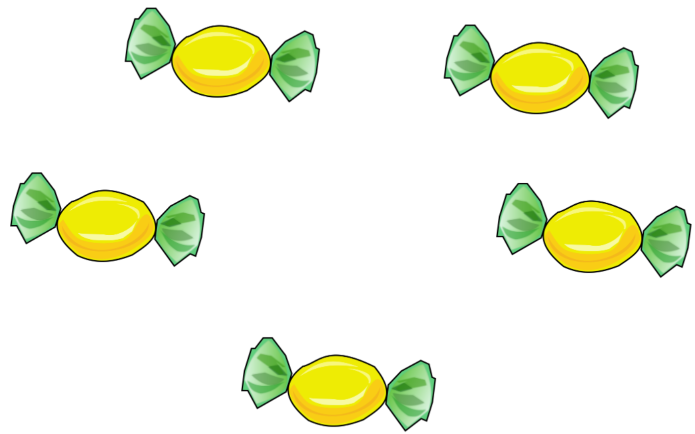
$$= (|A_1| + \cdots + |A_n|) - (|A_1 \cap A_2| + \cdots + |A_{n-1} \cap A_n|) + \cdots$$

Random Picture

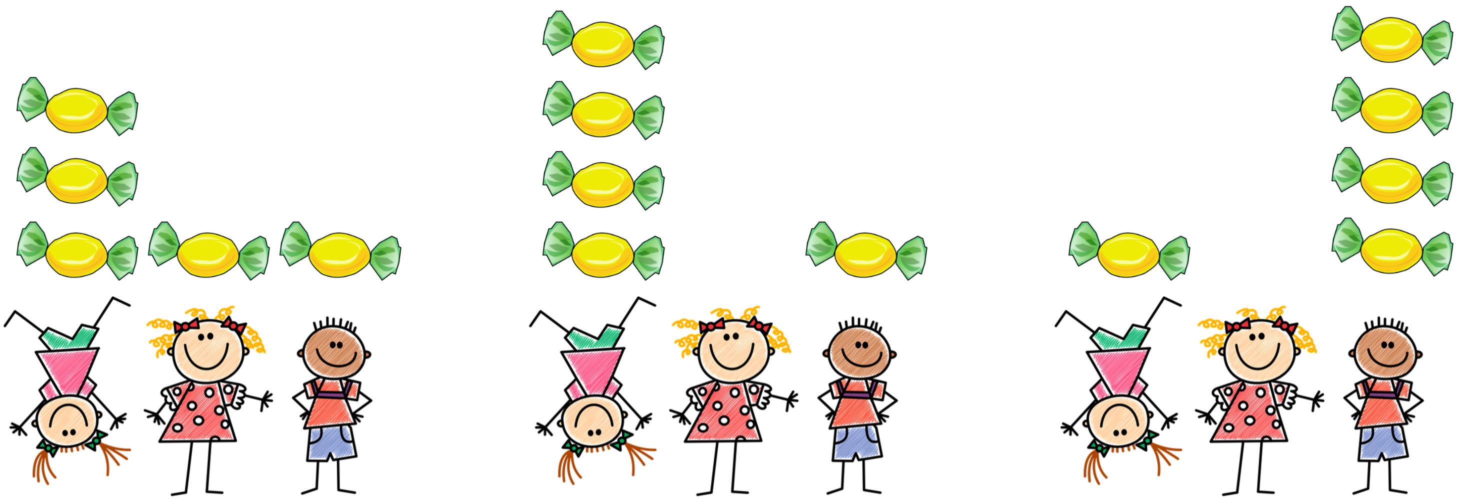
Stars & Bars

Kids + Candies

How many ways can we give 5 (Indistinguishable) candies to these 3 kids?



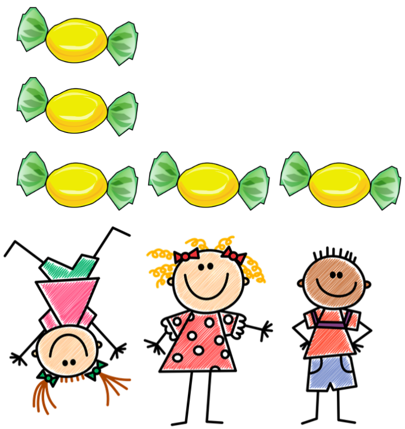
Kids + Candies



5 candies

6 → kid 1
→ kid 2

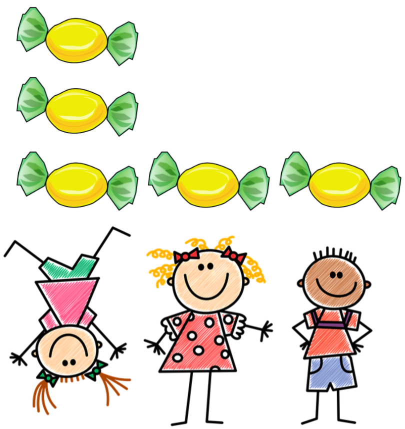
Kids + Candies



Idea: Count something equivalent.

5 "Stars" for candies, 2 "bars" for dividers

Kids + Candies



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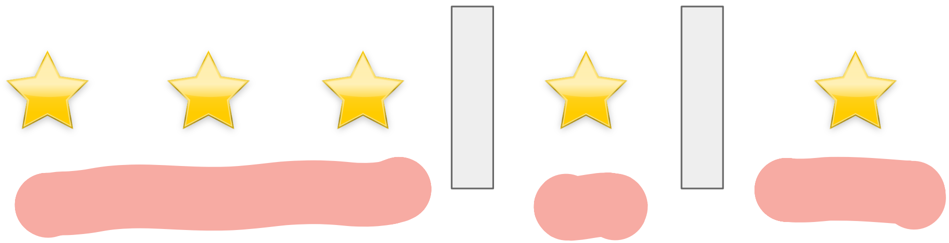
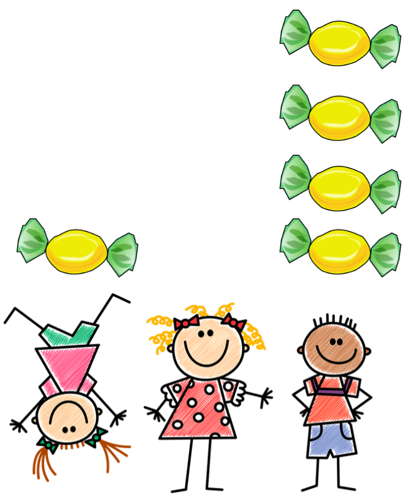


Kids + Candies



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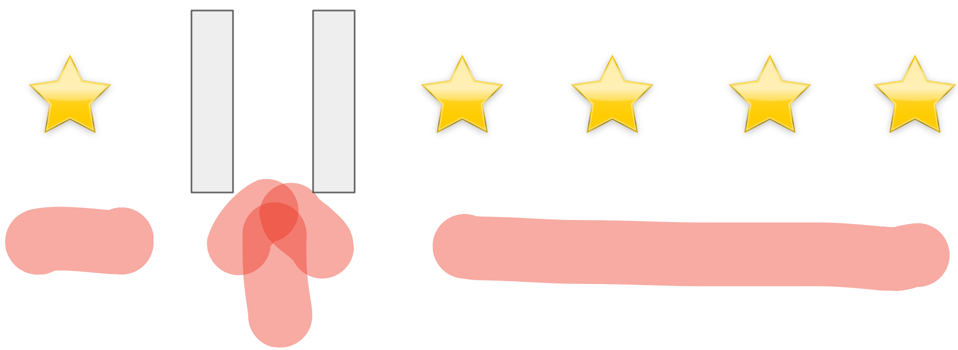
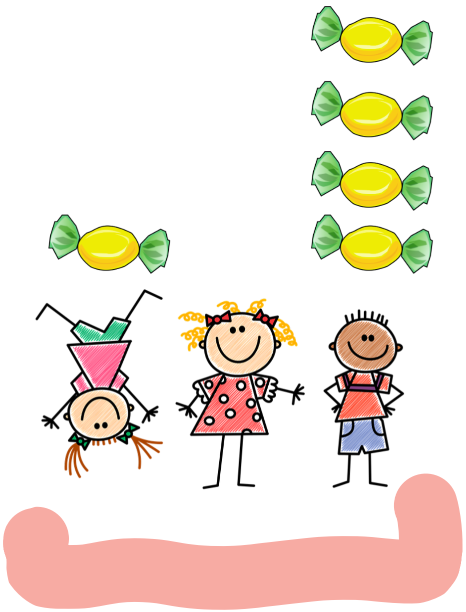
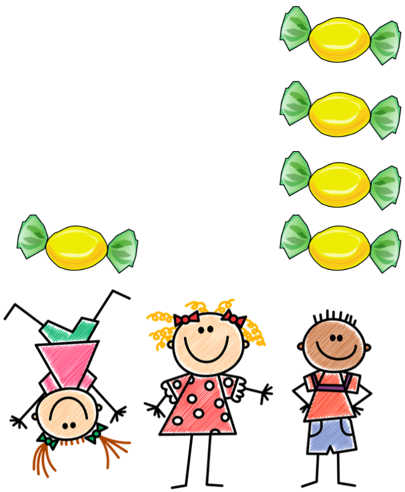


Kids + Candies



FOR EACH CANDY DISTRIBUTION, THERE IS EXACTLY ONE CORRESPONDING WAY TO ARRANGE THE STARS AND BARS.

Kids + Candies



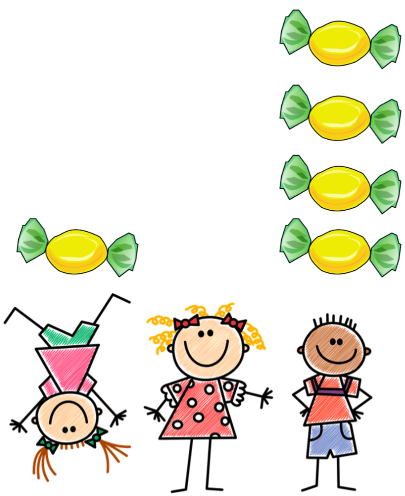
For each candy distribution, there is exactly one corresponding way to arrange the stars and bars.

Conversely, for each arrangement of stars and bars, there is exactly one candy distribution it represents.

Kids + Candies

$\star \;|\; |\; \star \; \star \; \star \; \star$

Hence, the number of ways to distribute 5 candies to the 3 kids is the number of arrangements of 5 stars and 2 bars.



How many ways?

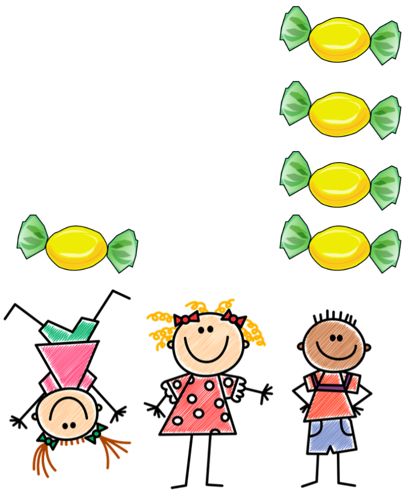
(a) $\binom{5}{2}$

(b) $\binom{7}{2}$

(c) $\binom{7}{5}$

(d) I don't know

Kids + Candies



Hence, the number of ways to distribute 5 candies to the 3 kids is the number of arrangements of 5 stars and 2 bars.

This is simply

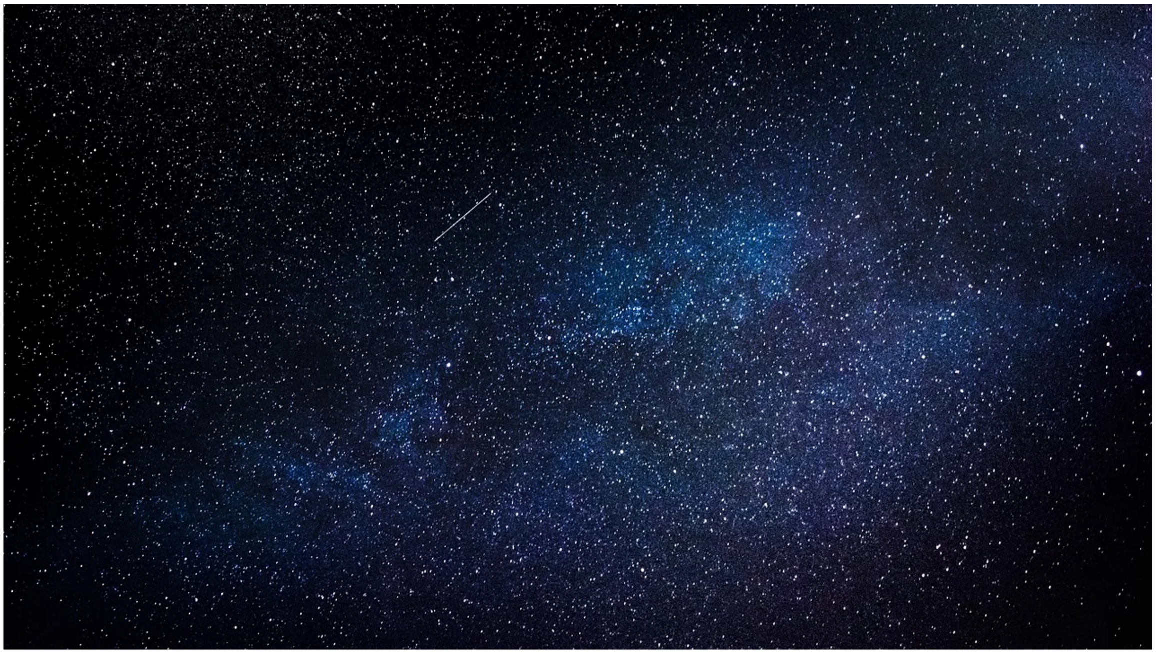
$$\binom{7}{2} = \binom{7}{5}$$

Stars and Bars/Divider method

The number of ways to distribute n indistinguishable balls into k distinguishable bins is

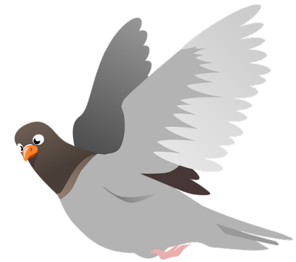
$$\binom{n+(k-1)}{k-1} = \binom{n+(k-1)}{n}$$

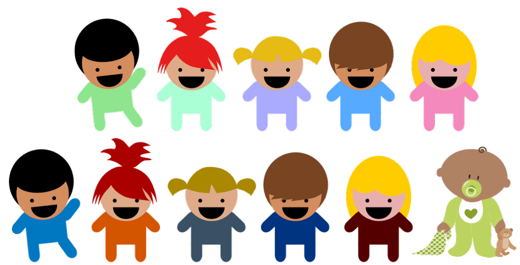
WE'LL BE COUNTING STARS (AND BARS)



Pigeonhole principle

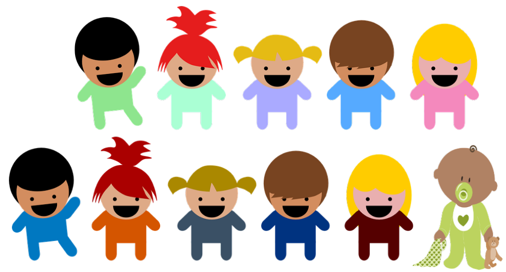
Pigeonhole Principle: idea





Suppose we split 11 children up into 3 groups and each group gets a cake to share. What is the largest number of children that will need to share a cake?

Pigeonhole Principle: idea



If 11 children have to share 3 cakes, at least one cake must by at least how many children? 4 (11/3 but rounded up)

Pigeonhole principle (PHP)

If there are n pigeons we want to put into k pigeonholes (where $n > k$), then at least one pigeonhole must contain at least 2 pigeons.

More generally, if there are n pigeons we want to put into k pigeonholes, then at least one pigeonhole must contain at least $\lceil n/k \rceil$ pigeons.

The floor and ceiling functions

The floor function $\lfloor x \rfloor$ returns the largest integer $\leq x$ (i.e., rounds down).

$$\lfloor 2.5 \rfloor = 2 \qquad \lfloor 16.99999 \rfloor = 16 \qquad \lfloor 5 \rfloor = 5$$

The ceiling function $\lceil x \rceil$ returns the smallest integer $\geq x$ (i.e., rounds up).

$$\lceil 2.5 \rceil = 3 \qquad \lceil 9.000301 \rceil = 10 \qquad \lceil 5 \rceil = 5$$

Pigeonhole principle (PHP)

If there are n pigeons we want to put into k pigeonholes (where $n > k$), then at least one pigeonhole must contain at least 2 pigeons.

More generally, if there are n pigeons we want to put into k pigeonholes, then at least one pigeonhole must contain at least $\lceil n/k \rceil$ pigeons.

Use the PHP to show that in every set of 100 numbers, there are two whose difference is a multiple of 37.

Pigeonhole principle (PHP)

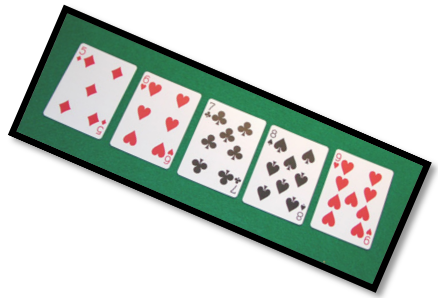
Use the PHP to show that in every set of 100 numbers, there are two whose difference is a multiple of 37.

When solving a PHP problem:

- Identify the Pigeons
- Identify the pigeonholes
- Specify how pigeons are assigned to holes
- Apply the principle

Let's practice some more

Quick review of cards



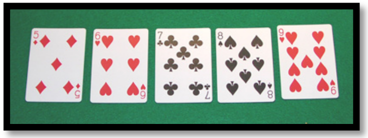
52 total cards

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

Counting cards

- How many possible 5 card hands?

- A "straight" is five consecutive rank cards of any suit. How many possible straights?



- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

Counting cards

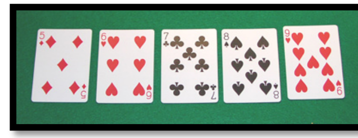
- How many possible 5 card hands?

$$\binom{52}{5}$$

- A flush is five card hand all of the same suit. How many possible fluhes?

Counting cards

A "straight" is five consecutive rank cards of any suit. How many possible straights?



$$10 \cdot 4^5 = 10,240$$

A flush is five card hand all of the same suit. How many possible flushes?

$$4 \cdot \binom{13}{5} = 5,148$$

How many flushes are not straights?

The Sleuth's criterion (Rudich)

For each object constructed, it should be possible to reconstruct the unique sequence of choices that led to it.

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

$$\binom{4}{3} \cdot \binom{49}{2}$$

First choose 3 Aces, then choose remaining two cards.

The Sleuth's criterion (Rudich)

For each object constructed, it should be possible to reconstruct the unique sequence of choices that led to it.

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

$\binom{4}{3} \cdot ($ … rst choose 3 Aces, then choose

… ining two cards.

The Sleuth's criterion (Rudich)

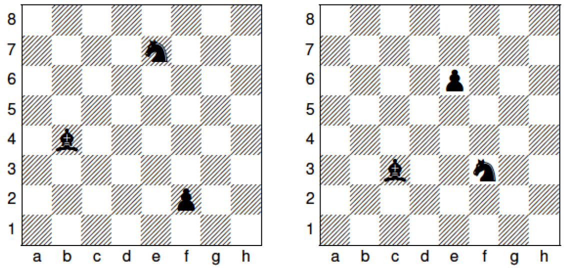
For each object constructed, it should be possible to reconstruct the unique sequence of choices that led to it.

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

When in doubt, break set up into disjoint sets you know how to count and then use the sum rule.

8 by 8 chessboard

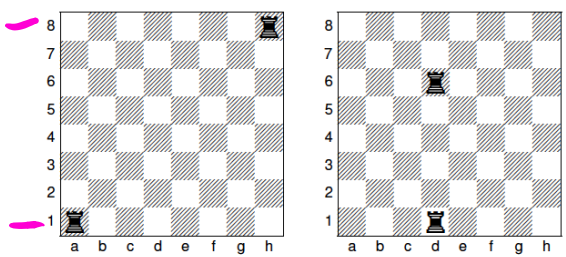
How many ways to place a pawn, a bishop and a knight so that none are in the same row or column



(a) valid (b) invalid

Rooks on chessboard

How many ways to place two identical rooks on a chessboard so that they don't share a row or a column



(a) valid (b) invalid

(a) $8^2 \cdot 7^2$

(b) $\binom{8}{2}\binom{8}{2}$

(c) $\frac{8^2 \cdot 7^2}{2}$

(d) I don't know

Doughnuts

You go to top pot to buy a dozen donuts. Your choices are

Chocolate, Lemon-filled, Maple, Glazed, plain

How many ways are there to choose a dozen doughnuts when doughnuts of the same type are indistinguishable?

Stars and Bars/Divider method

The number of ways to distribute n indistinguishable balls into k distinguishable bins is

$$\binom{n+(k-1)}{k-1} = \binom{n+(k-1)}{n}$$

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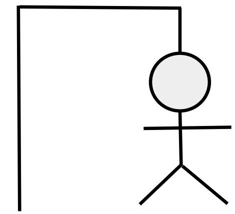
Doughnuts

You go to top pot to buy a dozen donuts. Your choices are

Chocolate, Lemon-filled, sugar, Glazed, plain

How many ways are there to choose a dozen doughnuts when you want at least 1 of each type?

ANAGRAMS



How many ways can you arrange the letters in "Godoggy"?

$N=7$ letters, $K=4$ types {G, O, D, Y}

$N_1 = 3, N_2 = 2, N_3 = 1, N_4 = 1$

$$\frac{7!}{3!\,2!\,1!\,1!} = \binom{7}{3, 2, 1, 1}$$

Multinomial Coefficients

If we have k types of objects (n total), with n_1 of the first type, n_2 of the second, ..., and n_k of the k^{th}, then the number of arrangements possible is

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!\, n_2!\, \ldots\, n_k!}$$

COMBINATORIAL ARGUMENT/PROOF

- LET S BE A SET OF OBJECTS
- SHOW HOW TO COUNT |S| ONE WAY => |S|=N
- SHOW HOW TO COUNT |S| ANOTHER WAY =>. |S| = M

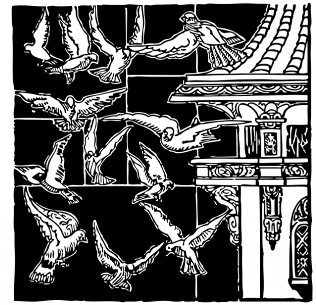
CONCLUDE N=M

$$\binom{n}{r} = \binom{n}{n-r}$$

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

$$\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$$

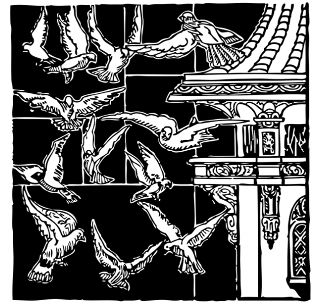
Combinatorial Proofs: example



Show that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Consider the set of numbers $\{1, 2, \ldots, n\}$.

Combinatorial Proofs: example



Show that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

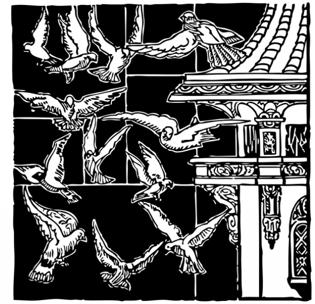
Consider the set of numbers $\{1,2,\dots,n\}$.

Left Side: Counts the number of subsets of size k.

Right Side: Two cases. We either include the number 1 or not.

- If we include the number 1, we need to choose $k-1$ out of the remaining $n-1$.
- If we don't include it, we need to choose k out of the remaining $n-1$.

Combinatorial Proofs: example



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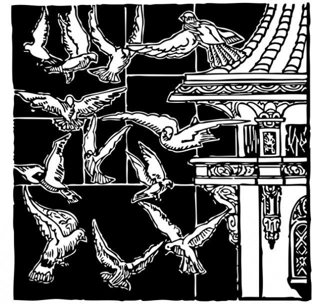

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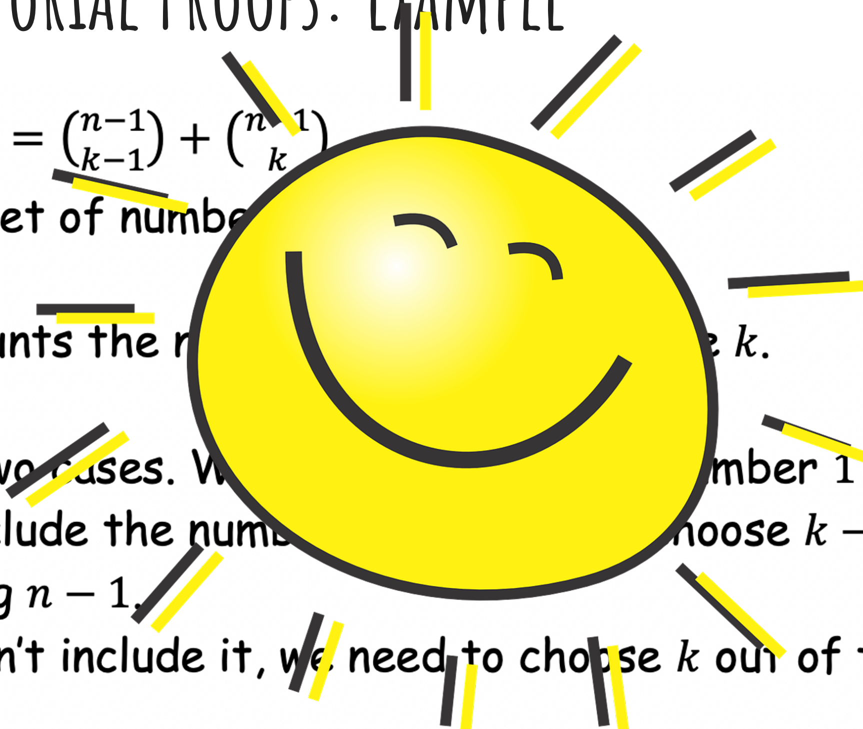
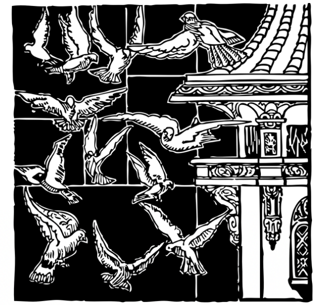


The alternative….



Show that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-1-k)!}$$
$$= 20\ years\ later\ \ldots$$
$$= \frac{n!}{k!\,(n-k)!}$$
$$= \binom{n}{k}$$

Tools and concepts

- Sum rule, Product rule
- Permutations, combinations
- Inclusion-exclusion
- Binomial Theorem
- Combinatorial proofs
- Pigeonhole principle
- Stars and bars

Counting is NOT for kindergarteners