Welcome to cse312!

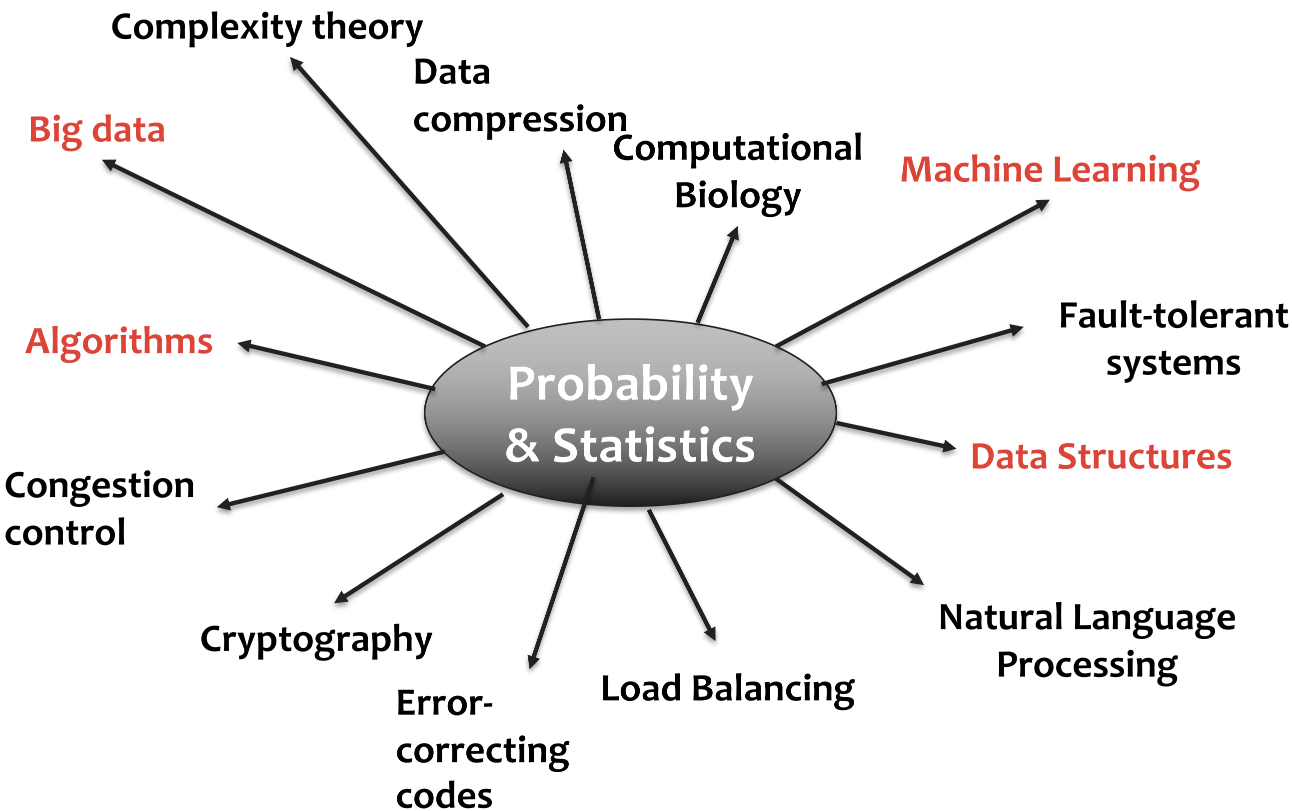
Slide design by Alex Tsun

Most important source of information

`http://courses.cs.washington.edu/312`

Why probability and statistics?

+ much more!


Complexity theory
Data compression
Big data
Computational Biology
Machine Learning
Algorithms
Probability & Statistics
Fault-tolerant systems
Data Structures
Congestion control
Cryptography
Error-correcting codes
Load Balancing
Natural Language Processing

1. counting

Anna Karlin

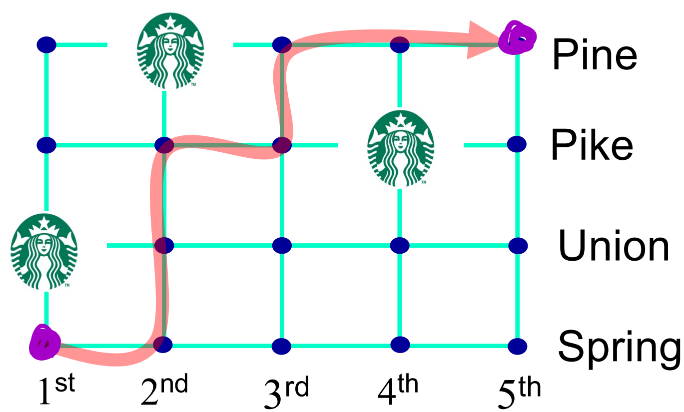
Most of the Slides created by Alex Tsun

Counting Is hard with only 10 fingers

How many ways are there to do X?

- X = “Choose a number between 1 and 10”

- X = “Walk from 1st and Spring to 5th and Pine



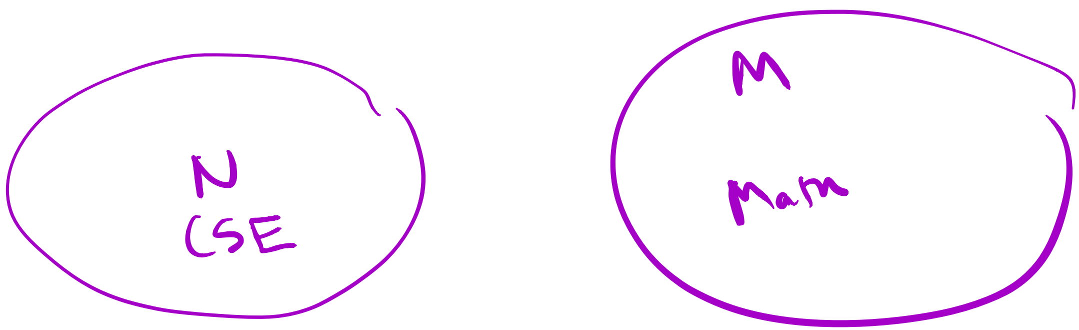
Counting is hard when numbers or large or constraints are complex

- Sum rule
- Product Rule
- Permutations
- Complementary Counting

Sum rule

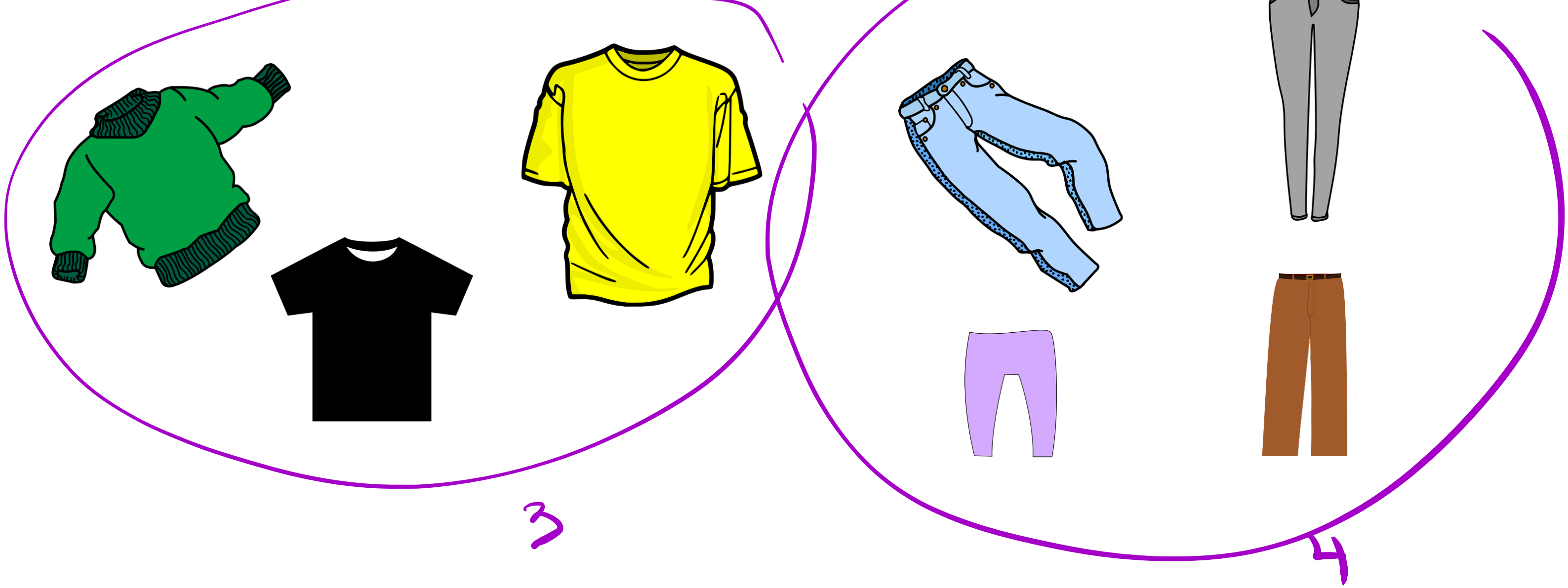
If an experiment can either end up being one of N outcomes, or one of M outcomes (where there is no overlap), then the number of possible outcomes of the experiment is

$$N+M$$



Counting "outfits"

If an outfit consists of **either** a top **or** a bottom, how many outfits are possible?

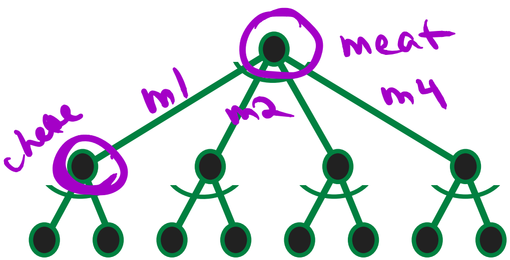


Product rule

If an experiment has N_1 outcomes for the first stage, N_2 outcomes for the second stage (given the first), …, and N_M outcomes for the M^{th} stage (given the previous stages), then the total number of outcomes of the experiment is: $N_1 \times N_2 \times \ldots N_M$

meat 4

cheese 2



$4 \cdot 2$

Product rule

If an experiment has N_1 outcomes for the first stage, N_2 outcomes for the second stage, given the first, …, and N_M outcomes for the M^{th} stage, given the previous, then the total number of outcomes of the experiment is. $N_1 \times N_2 \times \ldots N_M$

Example: How many n bit strings are there?

$$\boxed{2} \times \boxed{2} \times \boxed{2} \times \ldots \times \boxed{2} \times \boxed{2} = \boxed{2^N}$$

choices for 1st bit

choices for 2nd

Product Rule

- How many N-bit numbers are there? 2^N

- How many subsets of a set of size 26? {A,B,C,D,…,Z}

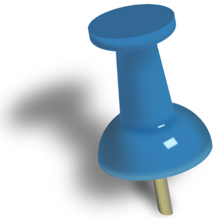
$$\boxed{2} \times \boxed{2} \times \boxed{2} \times \ldots \times \boxed{2} \times \boxed{2} = \boxed{2^{26}}$$

is A in or out

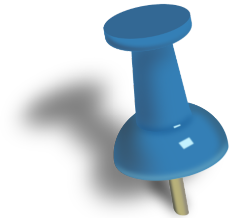
is B in or out

is Z in or not

ATM's and Pin Codes and robbers



ATM's and Pin Codes



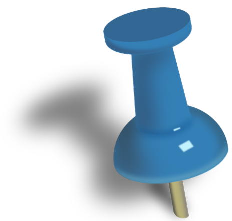
- How many 4-digit pin codes are there?
- Each digit is one of {0,1,2,…,9}

10 possible choices for each digit of pin code

$$\boxed{10} \times \boxed{10} \times \boxed{10} \times \boxed{10} = \boxed{10^4}$$

# possible first digits	# possible second digits	# possible third digits	# possible fourth digits	# possible 4-digit PINs

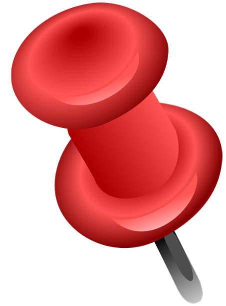
(Back to) ATM's and Pin Codes and robbers



10	x	10	x	10	x	10	=	10000
# possible first digits		# possible Second digits		# possible third digits		# possible fourth digits		# possible 4-Digit PINs

So one in ten thousand chance that a robber can guess your pin code...

Stronger pins



10-digit pin code, must use each digit **exactly once**

How many such pins?

$$\boxed{10} \times \boxed{9} \times \boxed{8} \times \ldots \times \boxed{2} \times \boxed{1} = \boxed{3,628,800}$$

# possible first digits	# possible second digits	# possible third digits	# possible ninth digits	# possible tenth digits	# possible PINs

$10!$

Permutations

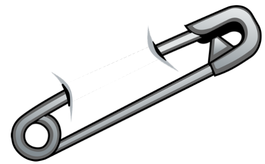
The number of orderings of n **distinct** objects is

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$$

Read as "n factorial"

26!

Tricky pins

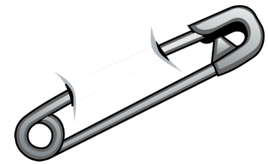


10-digit pin code, has at least one digit repeated at least once

How many such pins?

E.g., 1111111111, 0123456788, 9876598765, ...

Tricky pins

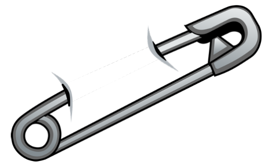


10-digit PIN code, has at least one digit repeated at least once

How many such pins?

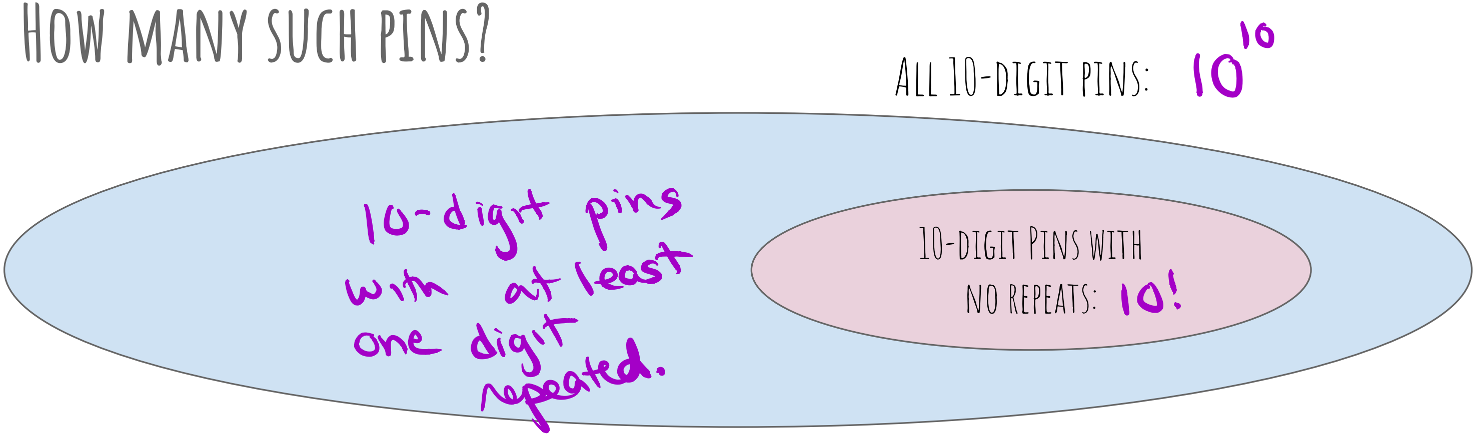
10	x	10	x		x … x		x		=	
# possible first digits		# possible second digits		# possible third digits		# possible ninth digits		# possible tenth digits		# possible PINs

Tricky pins



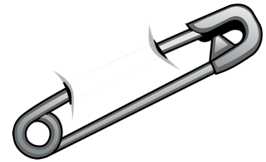
10-digit pin code, has at least one digit repeated at least once

How many such pins?



$10^{10} - 10!$

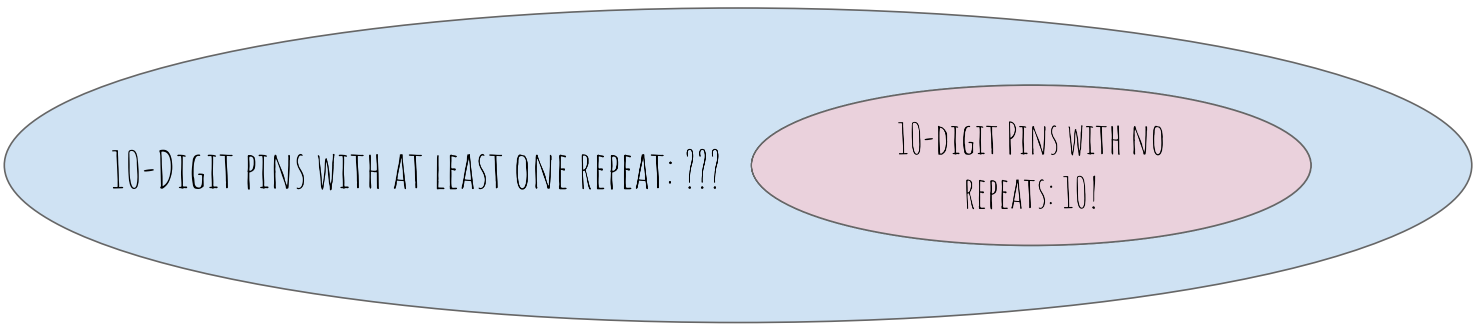
Tricky pins



10-digit pin code, has at least one digit repeated at least once

How many such pins? $\boxed{10^{10} - 10!}$

All 10-digit pins: 10^{10}



Complementary counting

Let U be a (finite) universal set, and S a subset of interest. Let $U \setminus S$ denote the set difference. Then,

$$|S| = |U| - |U \setminus S|$$

That is, the complement of the subset of interest is ALSO of interest...

"at least"

"some"

Let's expand our toolbox some more

- k-Permutations
- Combinations/Binomial Coefficients
- Combinatorial proofs
- Multinomial Coefficients

palate

Mini-Rainbows?

How many 3-Color Mini-Rainbows can be made out of 7 available colors?

MINI-RAINBOWS

HOW MANY 3-COLOR MINI-RAINBOWS CAN BE MADE OUT OF 7 AVAILABLE COLORS?

$$\boxed{7} \times \boxed{6} \times \boxed{5} = \boxed{210}$$

# POSSIBLE OUTER COLORS	# POSSIBLE MIDDLE COLORS	# POSSIBLE INNER COLORS	# POSSIBLE MINI-RAINBOWS

$$7 \cdot 6 \cdot 5$$

LET'S FIND A SHORTHAND

$$\underbrace{7 \times 6 \times 5} = \underbrace{7 \times 6 \times 5} \times \boxed{\frac{4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1}} = \frac{7!}{4!} = \frac{7!}{(7-3)!}$$

WE ARE "PICKING" 3 OUT OF THE 7 AVAILABLE COLORS.

Let's find a shorthand

$$7 \times 6 \times 5 = 7 \times 6 \times 5 \times \frac{4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} = \frac{7!}{4!} = \frac{7!}{(7-3)!}$$

We are "Picking" 3 out of the 7 available Colors.

K-Permutations

If we want to arrange **only** k out of n distinct objects, the number of ways to do so is

$$\boxed{P(n,k)} = \underline{n \times (n-1) \times (n-2) \times \ldots \times (n-k+1)} = \boxed{\frac{n!}{(n-k)!}}$$

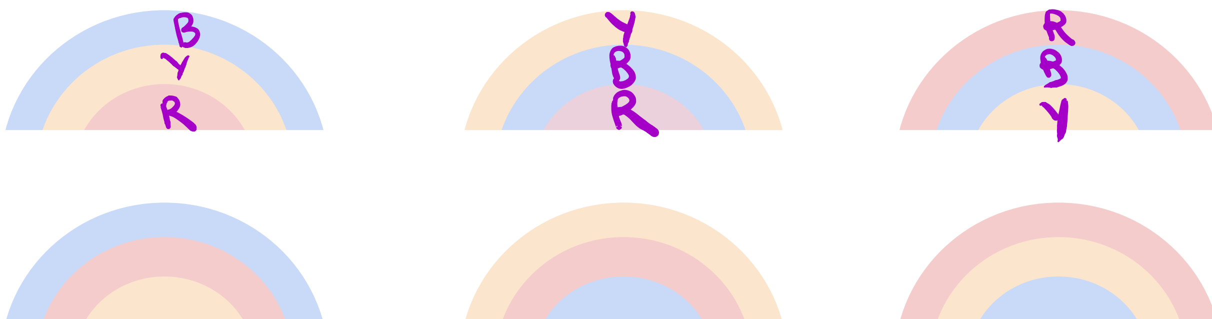
$n! = P(n,n)$

Read as "n pick k"

$2^{[n]}$

Smeared Mini-Rainbows

How many ways are there to pick 3 different colors out of 7 if I'm going to smear them all together?



1 smeared color $\Longrightarrow$ 3! mini rainbows

Mini-Rainbows Again

Recall there were 210 mini-rainbows. Look at these particular $3! = 6$ mini-rainbows with Blue, Orange, and Red.

They all produce the SAME "smear"!

Smeared vs mini-rainbows

\# "smeared" colors \# mini-rainbows

$$\underline{\underline{C(7,3)}} \cdot \underline{3!} = \underline{\underline{P(7,3)}}$$

$$\binom{7}{3} = C(7,3) = \frac{P(7,3)}{3!} = \frac{7!}{4!\,3!}$$

7 choose 3

Mini-Rainbows Again

Each "smeared" color is counted exactly 3! = 6 times, so we can take our 210 mini-rainbows and divide by 6 to get the answer!

$$\frac{P(7,3)}{3!} = \frac{7!}{3!(7-3)!}$$

Combinations/Binomial Coefficients

If we want to select (order doesn't matter) **only** k out of n distinct objects, the number of ways to do so is

$$C(n,k) = \binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}$$

Both Acceptable

Read as "n choose k"

Symmetry in Binomial Coefficient

Notice

$$\binom{N}{K} = \frac{N!}{K!(N-K)!} = \frac{N!}{(N-K)!K!} = \binom{N}{N-K}$$

$$\frac{P(N,K)}{K!} \qquad\qquad \frac{P(N,N-K)}{(N-K)!}$$

Symmetry in Binomial Coefficient

Notice $$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}$$

Why??

How??

More intuitive reason

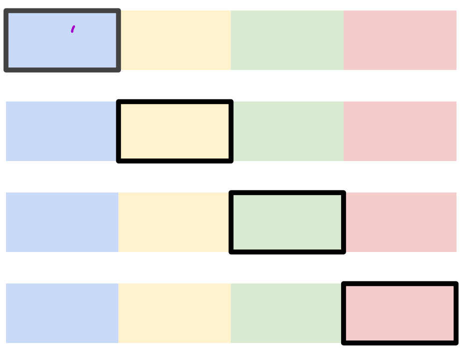
Number of ways to choose k out of n objects (unordered)

- Choose which k elements are included
- Choose which n-k elements are excluded

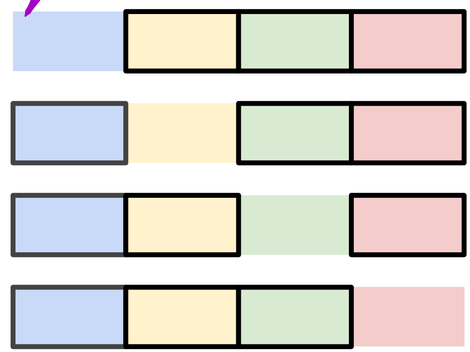
Symmetry in Binomial Coefficient

Consider n=4, k=1. We want to show C(4,1)=C(4,3) intuitively.

$C(4,1)=4=\binom{4}{1}$



$\binom{4}{3}= C(4,3)=4$



$$\binom{10}{3}=\binom{10}{7}$$

More intuitive reason

Number of ways to choose k out of n objects (unordered)

- Choose which k elements are included
- Choose which n-k elements are excluded

This is called a **combinatorial argument/proof**

- Let S be a set of objects
- Show how to count |S| one way => |S|=N
- Show how to count |S| another way => |S| = M

$\Rightarrow N = M$

$\binom{n}{k} = \binom{n}{n-k}$

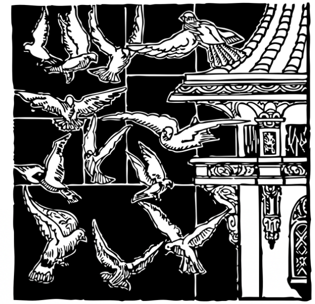
COMBINATORIAL ARGUMENT/PROOF

- Let S be a set of objects
- Show how to count |S| one way => |S|=N
- Show how to count |S| another way =>. |S| = M

CONCLUDE N=M

$$\binom{n}{r} = \binom{n}{n-r}$$
$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$
$$\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$$

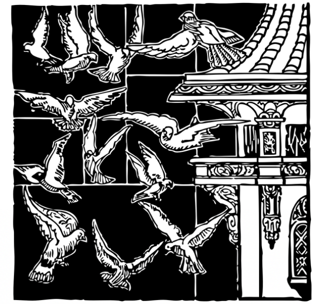
Combinatorial Proofs: Idea



Show that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Consider the set of numbers $\{1, 2, \ldots, n\}$.

Combinatorial Proofs: Idea

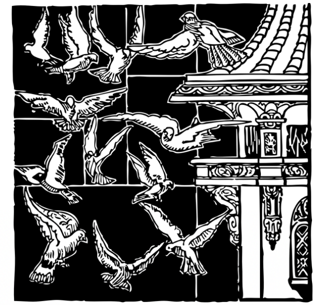


Show that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Consider the set of numbers $\{1,2,\ldots,n\}$.

Left Side: Counts the number of subsets of size k.

Combinatorial Proofs: Idea



Show that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

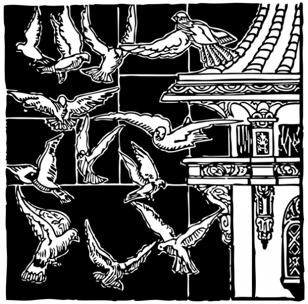
Consider the set of numbers $\{1,2,\ldots,n\}$.

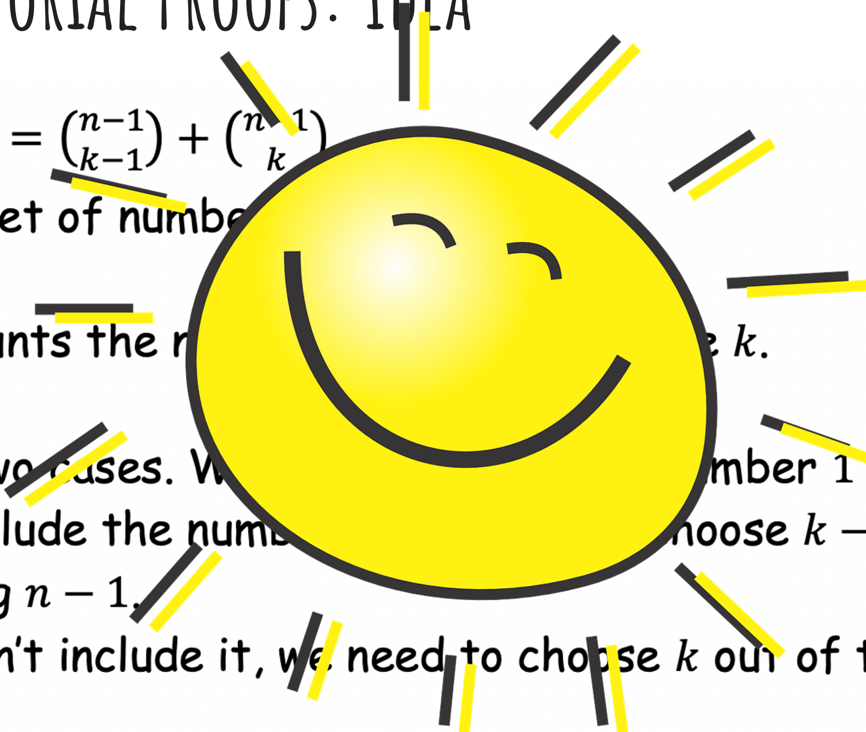
Left Side: Counts the number of subsets of size k.

Right Side: Two cases. We either include the number 1 or not.

- If we include the number 1, we need to choose $k-1$ out of the remaining $n-1$.
- If we don't include it, we need to choose k out of the remaining $n-1$.

Combinatorial Proofs: Idea





Show that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

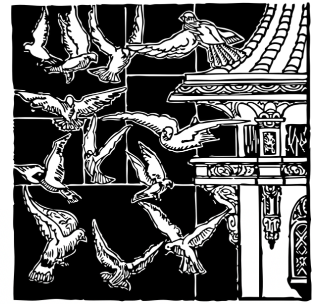
Consider the set of numbe

Left Side: Counts the n k.

Right Side: Two cases. W mber 1 or not.

- If we include the numb hoose $k-1$ out of the remaining $n-1$.
- If we don't include it, we need to choose k out of the remaining $n-1$.

The alternative….

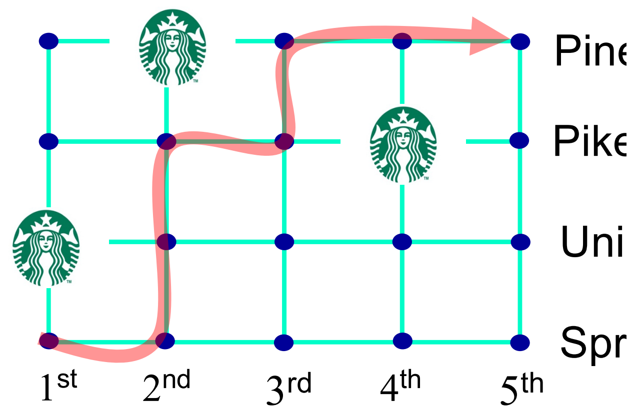


Show that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-1-k)!}$$
$$= 20\ \mathit{years\ later} \ldots$$
$$= \frac{n!}{k!\,(n-k)!}$$
$$= \binom{n}{k}$$

HOW MANY WAYS TO WALK FROM 1^{ST} AND SPRING TO 5^{TH} AND PINE?

ONLY GOING NORTH AND EAST



(a) 2^7

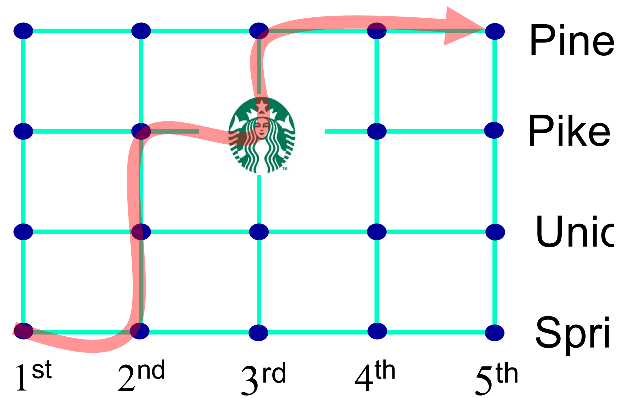
(b) $\binom{7}{3}$

(c) $\binom{7}{4}$

(d) $P(7,3)$

How many ways to walk from 1^{st} and spring to 5^{th} and pine, stopping at the starbucks on 3^{rd} and pike

Only going north and east



(a) $\binom{7}{3}$

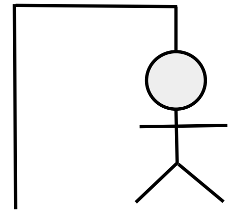
(b) $\binom{7}{3}\binom{7}{1}$

(c) $\binom{4}{2}\cdot\binom{3}{1}$

(d) $\binom{4}{2}\cdot\binom{3}{2}$

Random Picture

ANAGRAMS



HOW MANY WAYS CAN YOU ARRANGE THE LETTERS IN "MATH"?

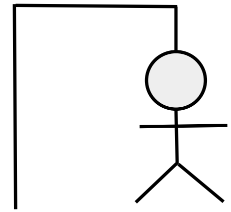
(a) 4^4

(b) $4!$

(c) $\binom{26}{4}$

MATH

ANAGRAMS

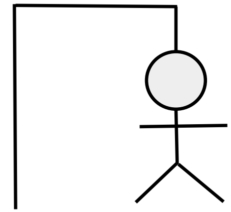


HOW MANY WAYS CAN YOU ARRANGE THE LETTERS IN "MATH"?

4! = 24 SINCE THEY ARE DISTINCT OBJECTS!

MATH

ANAGRAMS



How many ways can you arrange the letters in "MUUMUU"?

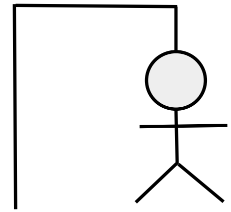
(a) $6!$

(b) $\binom{6}{2}\cdot\binom{4}{4}$

(c) $\binom{6}{4}$

(d) $6\cdot 5$

ANAGRAMS

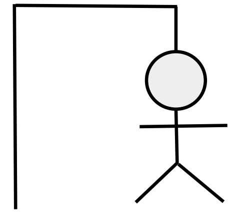


HOW MANY WAYS CAN YOU ARRANGE THE LETTERS IN "MUUMUU"?

CHOOSE WHERE THE 2 M'S GO, AND THEN THE U'S ARE SET. OR CHOOSE WHERE THE 4 U'S GO, AND THEN THE M'S ARE SET.

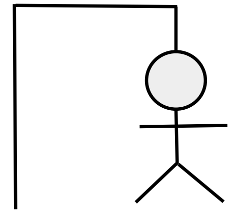
EITHER WAY, WE GET $\binom{6}{2} \cdot \binom{4}{4} = \binom{6}{4} \cdot \binom{2}{2} = \frac{6!}{2!4!}$

Another way to think about it



How many ways can you arrange the letters in "MUUMUU"?

ANAGRAMS



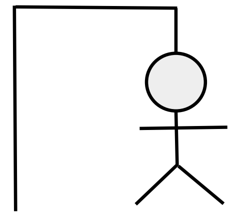
How many ways can you arrange the letters in "Muumuu"?

$$\binom{6}{2} \cdot \binom{4}{4} = \binom{6}{4} \cdot \binom{2}{2} = \frac{6!}{2!4!}$$

Another interpretation:

Arrange the 6 letters as if they were distinct. Then divide by 4! and 2! to account for 4 duplicate O's and 2 duplicate P's.

ANAGRAMS



How many ways can you arrange the letters in "Godoggy"?

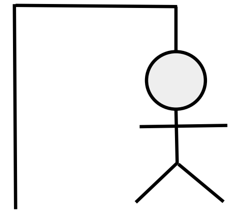
(a) $7!$

(b) $\frac{7!}{3!}$

(c) $\frac{7!}{3!\,2!\,1!}$

(d) $\binom{7}{3}\binom{5}{2}3!$

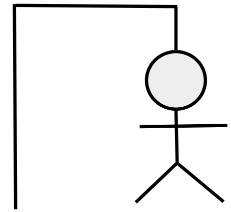
ANAGRAMS



HOW MANY WAYS CAN YOU ARRANGE THE LETTERS IN "GODOGGY"?

N=7 LETTERS, K=4 TYPES {G, O, D, Y}

ANAGRAMS



How many ways can you arrange the letters in "Godoggy"?

$N = 7$ letters, $K = 4$ types {G, O, D, Y}

$N_1 = 3, N_2 = 2, N_3 = 1, N_4 = 1$

$$\frac{7!}{3!\,2!\,1!\,1!} = \binom{7}{3, 2, 1, 1}$$

Multinomial Coefficients

If we have k types of objects (n total), with n_1 of the first type, n_2 of the second, ..., and n_k of the k^{th}, then the number of arrangements possible is

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!\, n_2! \ldots n_k!}$$

And finally...

- Binomial Theorem
- Inclusion-Exclusion
- Pigeonhole principle
- Stars and Bars/Divider Method

Binomial theorem: idea

$$(x + y)^2 = (x + y)(x + y)$$

$$xx + xy + yx + yy$$

$$x^2 + 2xy + y^2$$

Binomial theorem: idea

$$(x+y)^4 = (x+y)(x+y)(x+y)(x+y)$$

Binomial theorem: idea

$$(x+y)^4 = (x+y)(x+y)(x+y)(x+y)$$

$$xxxx + yyyy + xyxy + yxyy + \cdots$$

Binomial theorem: idea

$$(x+y)^4 = (x+y)(x+y)(x+y)(x+y)$$

Each term is of the form $x^k y^{n-k}$ (in our case, $n = 4$), since we multiply exactly n variables, either x or y.

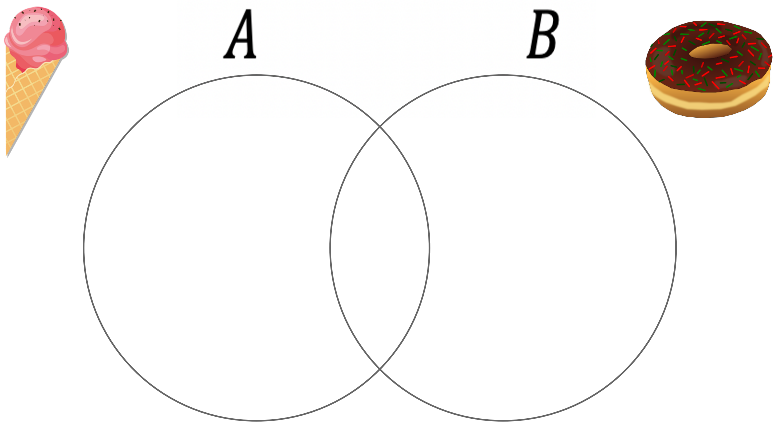
How many times do we get $x^k y^{n-k}$? The number of ways to choose k of them to produce x (the rest will be y).

Binomial Theorem

Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

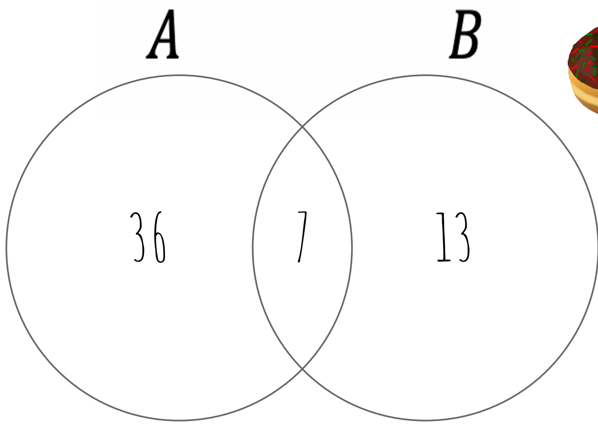
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Inclusion-Exclusion: idea



$$|A| = 43$$
$$|B| = 20$$
$$|A \cap B| = 7$$
$$|A \cup B| = ???$$

Inclusion-Exclusion: idea

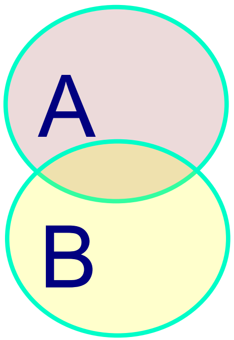


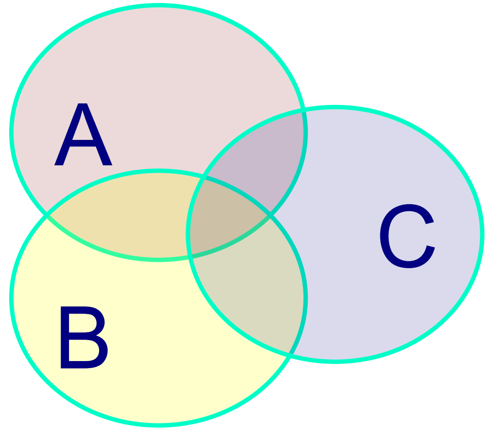
$$|A| = 43$$
$$|B| = 20$$
$$|A \cap B| = 7$$
$$|A \cup B| = ???$$

How many people like ice cream or donuts?

$$|A \cup B| = 36 + 7 + 13 = 56 = 43 + 20 - 7 = |A| + |B| - |A \cap B|$$

Inclusion-Exclusion





$|A \cup B| = |A| + |B| - |A \cap B|$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Inclusion-Exclusion

Let A, B be sets. Then,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

In general, if $A_1, A_2, \ldots, A_n$ are sets, then

$$|A_1 \cup \ldots \cup A_n| = singles - doubles + triples - quads + \cdots$$

$$= (|A_1| + \cdots + |A_n|) - (|A_1 \cap A_2| + \cdots + |A_{n-1} \cap A_n|) + \cdots$$