CSE 312: Foundations of Computing II

Section 8: Joint Distributions, Law of Total Expectation (and bit of conditional distributions) Solutions

1. Review of Main Concepts

(a) **Multivariate: Discrete to Continuous:**

	Discrete	**Continuous**
Joint PMF/PDF	$p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$	$f_{X,Y}(x,y) \neq \mathbb{P}(X = x, Y = y)$
Joint range/support $\Omega_{X,Y}$	$\{(x,y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x,y) > 0\}$	$\{(x,y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x,y) > 0\}$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t \leq x, s \leq y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\, ds dt$
Normalization	$\sum_{x,y} p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx dy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$
Expectation	$\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y) p_{X,Y}(x,y)$	$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$
Independence must have	$\forall x,y, p_{X,Y}(x,y) = p_X(x) p_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$	$\forall x,y, f_{X,Y}(x,y) = f_X(x) f_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$

(b) **Law of Total Probability (r.v. version):** If X is a discrete random variable, then

$$\mathbb{P}(A) = \sum_{x \in \Omega_X} \mathbb{P}(A | X = x) p_X(x) \qquad \text{discrete } X$$

(c) **Law of Total Expectation (Event Version)**: Let X be a discrete random variable, and let events $A_1, ..., A_n$ partition the sample space. Then,

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid A_i] \, \mathbb{P}(A_i)$$

(d) **Conditional Expectation**: See table. Note that linearity of expectation still applies to conditional expectation: $\mathbb{E}[X + Y \mid A] = \mathbb{E}[X \mid A] + \mathbb{E}[Y \mid A]$

(e) **Law of Total Expectation (RV Version)**: Suppose X and Y are random variables. Then,

$$\mathbb{E}[X] = \sum_{y} \mathbb{E}[X \mid Y = y] \, p_Y(y) \qquad \text{discrete version.}$$

(f) **Conditional distributions (not covered in class)**

	Discrete	**Continuous**
Conditional PMF/PDF	$p_{X\|Y}(x\|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$	$f_{X\|Y}(x\|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
Conditional Expectation	$\mathbb{E}[X \mid Y = y] = \sum_x x p_{X\|Y}(x\|y)$	$\mathbb{E}[X \mid Y = y] = \int_{-\infty}^{\infty} x f_{X\|Y}(x\|y) dx$

(g) **The following have not been covered as of 11/19:**

- Law of Total Probability (continuous)

$$\mathbb{P}(A) = \int_{x \in \Omega_X} \mathbb{P}(A | X = x) f_X(x) dx$$

- Law of total expectation (continuous)

$$\mathbb{E}[X] = \int_{y \in \Omega_Y} \mathbb{E}[X \mid Y = y] \, f_Y(y) dy$$

2. Joint PMF's

Suppose X and Y have the following joint PMF:

X/Y	1	2	3
0	0	0.2	0.1
1	0.3	0	0.4

(a) Identify the range of X (Ω_X), the range of Y (Ω_Y), and their joint range ($\Omega_{X,Y}$).

Solution:

$\Omega_X = \{0, 1\}$, $\Omega_Y = \{1, 2, 3\}$, and $\Omega_{X,Y} = \{(0, 2), (0, 3), (1, 1), (1, 3)\}$

(b) Find the marginal PMF for X, $p_X(x)$ for $x \in \Omega_X$.

Solution:

$$p_X(0) = \sum_y p_{X,Y}(0, y) = 0 + 0.2 + 0.1 = 0.3$$

$$p_X(1) = 1 - p_X(0) = 0.7$$

(c) Find the marginal PMF for Y, $p_Y(y)$ for $y \in \Omega_Y$.

Solution:

$$p_Y(1) = \sum_x p_{X,Y}(x, 1) = 0 + 0.3 = 0.3$$

$$p_Y(2) = \sum_x p_{X,Y}(x, 2) = 0.2 + 0 = 0.2$$

$$p_Y(3) = \sum_x p_{X,Y}(x, 3) = 0.1 + 0.4 = 0.5$$

(d) Are X and Y independent? Why or why not?

Solution:

No, since a necessary condition is that $\Omega_{X,Y} = \Omega_X \times \Omega_Y$.

(e) Find $\mathbb{E}\left[X^3 Y\right]$.

Solution:

Note that $X^3 = X$ since X takes values in $\{0, 1\}$.

$$\mathbb{E}\left[X^3 Y\right] = \mathbb{E}[XY] = \sum_{(x,y) \in \Omega_{X,Y}} xy p_{X,Y}(x, y) = 1 \cdot 1 \cdot 0.3 + 1 \cdot 3 \cdot 0.4 = 1.5$$

3. Trinomial Distribution

A generalization of the Binomial model is when there is a sequence of n independent trials, but with three outcomes, where $\mathbb{P}(\text{outcome } i) = p_i$ for $i = 1, 2, 3$ and of course $p_1 + p_2 + p_3 = 1$. Let X_i be the number of times outcome i occurred for $i = 1, 2, 3$, where $X_1 + X_2 + X_3 = n$. Find the joint PMF $p_{X_1,X_2,X_3}(x_1, x_2, x_3)$ and specify its value for all $x_1, x_2, x_3 \in \mathbb{R}$.

Solution:

Same argument as for the binomial PMF:

$$p_{X_1,X_2,X_3}(x_1, x_2, x_3) = \binom{n}{x_1, x_2, x_3} \prod_{i=1}^{3} p_i^{x_i} = \frac{n!}{x_1!x_2!x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3}$$

where $x_1 + x_2 + x_3 = n$ and are nonnegative integers.

4. Do You "Urn" to Learn More About Probability?

Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let $X_i = 1$ if the i-th ball selected is white and let it be equal to 0 otherwise. Give the joint probability mass function of

(a) X_1, X_2

Solution:

Here is one way of defining the joint pmf of X_1, X_2

$$\mathbb{P}(X_1 = 1, X_2 = 1) = \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 1 \mid X_1 = 1) = \frac{5}{13} \cdot \frac{4}{12} = \frac{20}{156}$$

$$\mathbb{P}(X_1 = 1, X_2 = 0) = \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 0 \mid X_1 = 1) = \frac{5}{13} \cdot \frac{8}{12} = \frac{40}{156}$$

$$\mathbb{P}(X_1 = 0, X_2 = 1) = \mathbb{P}(X_1 = 0)\mathbb{P}(X_2 = 1 \mid X_1 = 0) = \frac{8}{13} \cdot \frac{5}{12} = \frac{40}{156}$$

$$\mathbb{P}(X_1 = 0, X_2 = 0) = \mathbb{P}(X_1 = 0)\mathbb{P}(X_2 = 0 \mid X_1 = 0) = \frac{8}{13} \cdot \frac{7}{12} = \frac{56}{156}$$

(b) X_1, X_2, X_3

Solution:

Instead of listing out all the individual probabilities, we could write a more compact formula for the pmf. In this problem, the denominator is always $P(13, k)$, where k is the number of random variables in the joint pmf. And the numerator is $P(5, i)$ times $P(8, j)$ where i and j are the number of 1s and 0s, respectively. In this case the number of 1s is $x_1 + x_2 + x_3$, and the number of 0s is $(1 - x_1) + (1 - x_2) + (1 - x_3)$. Then, we can write the pmf as follows:

$$p_{X_1,X_2,X_3}(x_1, x_2, x_3) = \frac{10!}{13!} \cdot \frac{5!}{(5 - x_1 - x_2 - x_3)!} \cdot \frac{8!}{(5 + x_1 + x_2 + x_3)!}$$

5. Successes

Consider a sequence of independent Bernoulli trials, each of which is a success with probability p. Let X_1 be the number of failures preceding the first success, and let X_2 be the number of failures between the first 2 successes. Find the joint pmf of X_1 and X_2. Write an expression for $E[\sqrt{X_1 X_2}]$. You can leave your answer in the form of a sum.

Solution:

X_1 and X_2 take on two particular values x_1 and x_2, when there are x_1 failures followed by one success, and then x_2 failures followed by one success. Since the Bernoulli trials are independent the joint pmf is

$$p_{X_1,X_2}(x_1,x_2) = (1-p)^{x_1}p \cdot (1-p)^{x_2}p = (1-p)^{x_1+x_2}p^2$$

By the definition of expectation

$$E[\sqrt{X_1X_2}] = \sum_{(x_1,x_2)\in\Omega_{X_1,X_2}} \sqrt{x_1x_2} \cdot (1-p)^{x_1+x_2}p^2$$

6. Continuous joint density I

The joint probability density function of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) & 0 < x < 1,\ 0 < y < 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Verify that this is indeed a joint density function.

(b) Compute the marginal density function of X.

(c) Find $Pr(X > Y)$. (Uses the continuous law of total probability which we have not covered in class as of 11/19.)

(d) Find $P(Y > \frac{1}{2} | X < \frac{1}{2})$.

(e) Find $E(X)$.

(f) Find $E(Y)$

Solution:

(a) A joint density function will integrate to 1 over all possible values. Since it $f_{X,Y}$ is 0 outside of the specified range, we can just integrate over that range using Wolfram Alpha, to get:

$$\int_0^2 \int_0^1 \frac{6}{7}(x^2 + \frac{xy}{2})dxdy = 1$$

(b) By definition, the marginal density function of X is the integration over all values of y, which again we can only go over the range of y to get:

$$f_X(x) = \int_0^2 \frac{6}{7}(x^2 + \frac{xy}{2})dy = \frac{6}{7}x(2x+1)$$

Note that this is only the case when $0 < x < 1$. Otherwise, the PDF is equal to 0.

(c) First, we rearrange our initial probability. Then, by the continuous law of total probability:

$$\mathbb{P}(X > Y) = 1 - \mathbb{P}(X \le Y) = 1 - \int_{-\infty}^{\infty} \mathbb{P}(X \le Y | Y = y) f_Y(y)dy = 1 - \int_{-\infty}^{\infty} \mathbb{P}(X \le y) f_Y(y)dy$$

Once again, we can instead integrate over just the range of y, getting:

$$1 - \int_0^2 \mathbb{P}(X \le y) f_Y(y)dy$$

We have to remember that x operates on the range $0 < x < 1$. Thus, the CDF is 1 past the top of that range, so we have:

$$1 - \int_0^1 \mathbb{P}(X \leq y) f_Y(y) dy - \int_1^2 f_Y(y) dy$$

So, now we just need to find the CDF of X, and the marginal PDF of Y. For the former:

$$F_X(x) = \int_0^x \frac{6}{7} u(2u+1) du = \frac{1}{7} x^2 (4x+3)$$

For the latter:

$$f_Y(y) = \int_0^1 \frac{6}{7} (x^2 + \frac{xy}{2}) dx = \frac{1}{14}(3y+4)$$

Putting these together, we get that:

$$\mathbb{P}(X > Y) = 1 - \int_0^1 \frac{1}{7} y^2 (4y+3) \frac{1}{14}(3y+4) dy - \int_1^2 \frac{1}{14}(3y+4) dy = 1 - \frac{253}{1960} - \frac{17}{28} = \frac{517}{1960} = 0.2638$$

(d) By the definition of conditional probability:

$$\mathbb{P}(Y > \frac{1}{2} | X < \frac{1}{2}) = \frac{\mathbb{P}(Y > \frac{1}{2}, X < \frac{1}{2})}{\mathbb{P}(X < \frac{1}{2})}$$

We can solve for the numerator by integrating over the given range for the joint distribution. We know that the joint PDF is 0 outside of its range, so we can truncate the integral to that:

$$\mathbb{P}(Y > \frac{1}{2}, X < \frac{1}{2}) = \int_{1/2}^{\infty} \int_{-\infty}^{1/2} f_{X,Y}(x,y) dx dy = \int_{1/2}^{\infty} \int_{-\infty}^{1/2} \frac{6}{7} \left(x^2 + \frac{xy}{2} \right) dx dy$$

$$= \int_{1/2}^{2} \int_{0}^{1/2} \frac{6}{7} \left(x^2 + \frac{xy}{2} \right) dx dy = \frac{69}{448}$$

For the denominator, we can integrate over the marginal distribution that we found before:

$$\int_{-\infty}^{1/2} \frac{6}{7} x(2x+1) dx = \frac{5}{28}$$

Putting these together, we get:

$$\mathbb{P}(Y > \frac{1}{2} | X < \frac{1}{2}) = \frac{\frac{69}{448}}{\frac{5}{28}} = 0.8625$$

(e) By definition, and because of the aforementioned range situation:

$$\mathbb{E}[X] = \int_0^1 f_X(x) x dx = \int_0^1 \frac{6}{7} x(2x+1) x dx = \frac{5}{7}$$

(f) By definition, and because of the aforementioned range situation:

$$\mathbb{E}[Y] = \int_0^2 f_Y(y) y dy = \int_0^2 \frac{1}{14}(3y+4) y dy = \frac{8}{7}$$

7. Continuous joint density II

The joint density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} xe^{-(x+y)} & x > 0, y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

and the joint density of W and V is given by

$$f_{W,V}(w,v) = \begin{cases} 2 & 0 < w < v, 0 < v < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Are X and Y independent? Are W and V independent?

Solution:

For two random variables X, Y to be independent, $f_{X,Y}(x,y) = f_X(x)f_Y(y)$. Let's start with X and Y by finding their marginal PDFs. By definition, the marginal density function of X is the integration over all values of y. Since the joint PDF is 0 outside of $y > 0$, we can integrate only over that range to get:

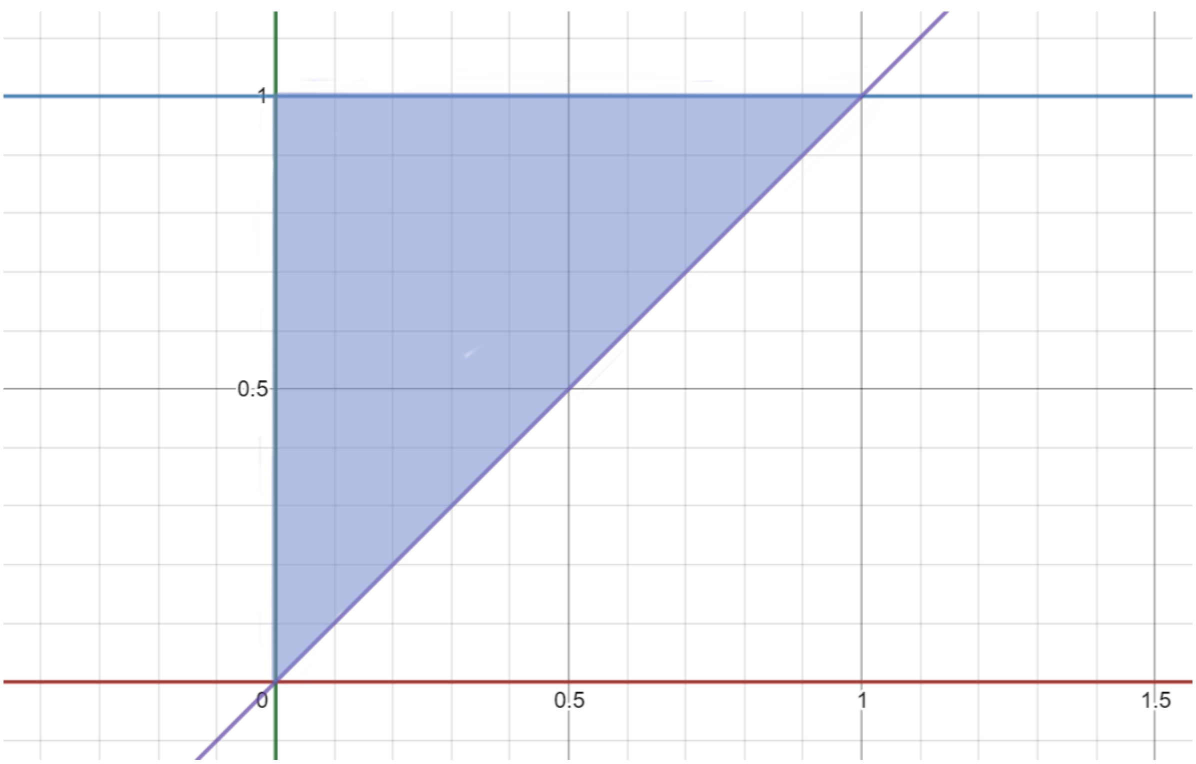
$$f_X(x) = \int_0^\infty xe^{-(x+y)}dy = e^{-x}x$$

We do the same to get the PDF of Y, with the same range of x: $x > 0$:

$$f_Y(y) = \int_0^\infty xe^{-(x+y)}dx = e^{-y}$$

Since $e^{-x}x \cdot e^{-y} = xe^{-x-y} = xe^{-(x+y)}$, X and Y are independent.

We can identify that W and V are not independent simply by looking at their range of $f_{W,V}(w,v)$. Graphing it with w as the "x-axis" and v as the "y-axis", we get:



The triangular space between the lines is the range. Looking at it, we can see that it is not rectangular, and therefore it is impossible that $\Omega_{W,V} = \Omega_W \times \Omega_V$. Remember, the joint range being the cartesian product of the marginal ranges is not sufficient for independence, but it is *necessary*. Therefore, this is enough to show that they are not independent.
We could have also shown the same process for W and V. We can start with the marginal PDF of W. Since the joint PDF is 0 outside of the range $0 < v < 1$, we can integrate over that range to get:

$$f_W(w) = \int_0^1 2dv = 2$$

We do the same to get the PDF of V, integrating with respect to w over the range $0 < w < v$:

$$f_V(v) = \int_0^v 2dw = 2v$$

Since $2 \cdot 2v \neq 2$, W and V are NOT independent.

8. Trapped Miner

A miner is trapped in a mine containing 3 doors.

- D_1: The 1^{st} door leads to a tunnel that will take him to safety after 3 hours.
- D_2: The 2^{nd} door leads to a tunnel that returns him to the mine after 5 hours.
- D_3: The 3^{rd} door leads to a tunnel that returns him to the mine after a number of hours that is Binomial with parameters $(12, \frac{1}{3})$.

At all times, he is equally likely to choose any one of the doors. What is the expected number of hours for this miner to reach safety?

Solution:

Let T = number of hours for the miner to reach safety. (T is a random variable)
Let D_i be the event the i^{th} door is chosen. $i \in \{1, 2, 3\}$. Finally, let T_3 be the time it takes to return to the mine in the third case only (a random variable). Note that the expectation of T_3 is $12 * \frac{1}{3}$ because it is binomially distributed with parameters $n = 12, p = 13$. By Law of Total Expectation, linearity of expectation, and by applying the conditional expectations given by the problem statement:

$$\begin{aligned}
\mathbb{E}[T] &= \mathbb{E}[T \mid D_1]\,\mathbb{P}(D_1) + \mathbb{E}[T \mid D_2]\,\mathbb{P}(D_2) + \mathbb{E}[T \mid D_3]\,\mathbb{P}(D_3) \\
&= 3 \cdot \frac{1}{3} + (5 + \mathbb{E}[T]) \cdot \frac{1}{3} + (\mathbb{E}[T_3 + T]) \cdot \frac{1}{3} \\
&= 3 \cdot \frac{1}{3} + (5 + \mathbb{E}[T]) \cdot \frac{1}{3} + (\mathbb{E}[T_3] + \mathbb{E}[T]) \cdot \frac{1}{3} \\
&= 3 \cdot \frac{1}{3} + (5 + \mathbb{E}[T]) \cdot \frac{1}{3} + (4 + \mathbb{E}[T]) \cdot \frac{1}{3}
\end{aligned}$$

Solving this equation for $\mathbb{E}[T]$, we get

$$\mathbb{E}[T] = 12$$

Therefore, the expected number of hours for this miner to reach safety is 12.

9. Lemonade Stand

Suppose I run a lemonade stand, which costs me \$100 a day to operate. I sell a drink of lemonade for \$20. Every person who walks by my stand either buys a drink or doesn't (no one buys more than one). If it is raining, n_1 people walk by my stand, and each buys a drink independently with probability p_1. If it isn't raining, n_2 people walk by my stand, and each buys a drink independently with probability p_2. It rains each day with probability p_3, independently of every other day. Let X be my profit over the next week. In terms of n_1, n_2, p_1, p_2 and p_3, what is $\mathbb{E}[X]$?

Solution:

Let R be the event it rains. Let X_i be how many drinks I sell on day i for $i = 1, ..., 7$. We are interested in $X = \sum_{i=1}^{7}(20X_i - 100)$. We have $X_i|R \sim \text{Binomial}(n_1, p_1)$, so $\mathbb{E}[X_i \mid R] = n_1 p_1$. Similarly, $X_i|R^C \sim \text{Binomial}(n_2, p_2)$, so $\mathbb{E}\left[X_i \mid R^C\right] = n_2 p_2$. By the law of total expectation,

$$\mu = \mathbb{E}[X_i] = \mathbb{E}[X_i \mid R]\,\mathbb{P}(R) + \mathbb{E}\left[X_i \mid R^C\right]\mathbb{P}(R^C) = n_1 p_1 p_3 + n_2 p_2 (1 - p_3)$$

Hence, by linearity of expectation,

$$\begin{aligned}\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{7}(20X_i - 100)\right] &= 20\sum_{i=1}^{7}\mathbb{E}[X_i] - 700 = 140\mu - 700 \\ &= 140 \cdot (n_1 p_1 p_3 + n_2 p_2 (1 - p_3)) - 700.\end{aligned}$$

10. Particle Emissions

Suppose we are measuring particle emissions, and the number of particles emitted follows a Poisson distribution with parameter λ, $X \sim \text{Poisson}(\lambda)$. Suppose our device to measure emissions is not always entirely accurate sometimes we fail to observe particles that actually emitted. So for each particle actually emitted, say we have probability p of actually recording it, independently of other particles. Let Y be the number of particles we observed. What distribution does Y follow with what parameters, and what is $\mathbb{E}[Y]$?

Solution:

$$\begin{aligned}
p_Y(y) &= \mathbb{P}(Y = y) \\
&= \sum_{x=y}^{\infty}\mathbb{P}(Y = y|X = x)\mathbb{P}(X = x) \quad \text{(Law of Total Probability)} \\
&= \sum_{x=y}^{\infty}\binom{x}{y}p^y(1-p)^{x-y} \cdot e^{-\lambda}\frac{\lambda^x}{x!} \quad \text{(Plug in Poisson and Binomial PMFs)} \\
&= e^{-\lambda}p^y\sum_{x=y}^{\infty}\frac{x!}{y!(x-y)!}(1-p)^{x-y}\frac{\lambda^x}{x!} \\
&= \frac{e^{-\lambda}p^y}{y!}\sum_{x=y}^{\infty}\frac{\lambda^x}{(x-y)!}(1-p)^{x-y} \\
&= \frac{e^{-\lambda}p^y}{y!}\sum_{k=0}^{\infty}\frac{\lambda^{k+y}}{k!}(1-p)^k \quad \text{(let } k = x - y\text{)} \\
&= \frac{e^{-\lambda}(\lambda p)^y}{y!}\sum_{k=0}^{\infty}\frac{(\lambda(1-p))^k}{k!} \\
&= \frac{e^{-\lambda}(\lambda p)^y}{y!} \cdot e^{\lambda(1-p)} \quad \text{(Taylor series for } e^{\lambda(1-p)}\text{)} \\
&= \frac{e^{-p\lambda}(\lambda p)^y}{y!}
\end{aligned} \tag{1}$$

So $Y \sim \text{Poisson}(p\lambda)$ and $\mathbb{E}[Y] = p\lambda$.

11. In between

(Covers ideas that have not been covered in class.) Suppose that X_1 and X_2 are discrete uniform random variables in $\{1, \ldots, 2n\}$ (i.e., X_1 and X_2 are equally likely to take any of the values $1, \ldots, 2n$) and let $Y = \min(X_1, X_2)$. What is the conditional pmf $p_{Y|X_1}(y \mid x_1)$ and conditional CDF $F_{Y|X_1}(y \mid x_1)$. What is $E[Y \mid X_1 = x_1]$? (For the definitions of conditional pmf, conditional CDF, see the review at the top of this worksheet.)

Solution:

The conditional pmf is

$$p_{Y|X_1}(y \mid x_1) = \mathbb{P}(\min(X_1, X_2) = y \mid X_1 = x_1) = \frac{\mathbb{P}(\min(X_1, X_2) = y, X_1 = x_1)}{\mathbb{P}(X_1 = x_1)} = \begin{cases} 0 & \text{if } y > x_1 \\ 1 - \frac{x_1 - 1}{2n} & \text{if } y = x_1 \\ \frac{1}{2n} & \text{if } 1 \le y < x_1 \end{cases}$$

Explanation for the final equality:

- Since $\min(X_1, X_2) \le X_1$, if $y > x_1$, then we have

$$\mathbb{P}(\min(X_1, X_2) = y, X_1 = x_1) = 0$$

- When $y = x_1$, we have

$$\mathbb{P}(\min(X_1, X_2) = x_1 \mid X_1 = x_1) = \mathbb{P}(X_2 \ge x_1 \mid X_1 = x_1) = \mathbb{P}(X_2 \ge x_1) = \frac{2n - x_1 + 1}{2n} = 1 - \frac{x_1 - 1}{2n}$$

  The second to last equality holds because X_1 and X_2 are independent.

- In the case where $y < x_1$, $(\min(X_1, X_2) = X_2$. So for $y < x_1$,

$$\mathbb{P}(Y = y \mid X_1 = x_1) = \mathbb{P}(X_2 = y \mid X_1 = x_1) = \mathbb{P}(X_2 = y) = \frac{1}{2n}.$$

  Again, we used the independence of X_1 and X_2.

The conditional CDF can be computed as follows

$$F_{Y|X_1}(y \mid x_1) = \sum_{i=1}^{y} p_{Y|X_1}(i \mid x_1) = \begin{cases} 0 & \text{if } y < 0 \\ \frac{y}{2n} & \text{if } y < x_1 \\ 1 & \text{if } y \ge x_1 \end{cases}$$

The expected value can be computed as follows:

$$E[Y \mid X_1 = x_1] = \sum_{y=1}^{2n} y \, p_{Y|X_1}(y \mid x_1) = \sum_{y=1}^{x_1 - 1} y \frac{1}{2n} + \sum_{y=x_1}^{x_1} y \left(1 - \frac{y-1}{2n}\right) = \frac{1}{2n} \sum_{y=1}^{x_1 - 1} y \; + \; x_1 \left(1 - \frac{x_1 - 1}{2n}\right)$$

12. 3 points on a line

(This problem uses the continuous law of total probability which has not yet be covered in class.) Three points X_1, X_2, X_3 are selected at random on a line L (continuous independent uniform distributions). What is the probability that X_2 lies between X_1 and X_3?

Solution:

Let $X_1, X_2, X_3 \sim Unif(0,1)$.

$$\begin{aligned}
\mathbb{P}(X_1 < X_2 < X_3) &= \int_{-\infty}^{\infty} \mathbb{P}(X_1 < X_2 < X_3 \mid X_2 = x)\, f_{X_2}(x)\, dx && \text{Continuous LoTP} \\
&= \int_{-\infty}^{\infty} \mathbb{P}(X_1 < x, X_3 > x)\, f_{X_2}(x)\, dx && \text{independence of } X_1, X_2, X_3 \\
&= \int_{-\infty}^{\infty} \mathbb{P}(X_1 < x)\, \mathbb{P}(x < X_3)\, f_{X_2}(x)\, dx && \text{Independence} \\
&= \int_{-\infty}^{\infty} F_{X_1}(x)\, (1 - F_{X_3}(x))\, f_{X_2}(x)\, dx \\
&= \int_{0}^{1} x\, (1 - x)\, 1\, dx \\
&= \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1 = \frac{1}{6}
\end{aligned}$$