CSE 312: Foundations of Computing II

Section 8: Joint Distributions, Law of Total Expectation (and bit of conditional distributions)

1. Review of Main Concepts

(a) Multivariate: Discrete to Continuous:

<table>
<thead>
<tr>
<th>Discrete</th>
<th>Continuous</th>
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</thead>
<tbody>
<tr>
<td>Joint PMF/PDF</td>
<td>$p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$</td>
</tr>
<tr>
<td>Joint range/support</td>
<td>${ (x,y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x,y) &gt; 0 }$</td>
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<tr>
<td>Joint CDF</td>
<td>$F_{X,Y}(x,y) = \sum_{t \leq x, s \leq y} p_{X,Y}(t,s)$</td>
</tr>
<tr>
<td>Normalization</td>
<td>$\sum_{x,y} p_{X,Y}(x,y) = 1$</td>
</tr>
<tr>
<td>Marginal PMF/PDF</td>
<td>$p_X(x) = \sum_{y} p_{X,Y}(x,y)$</td>
</tr>
<tr>
<td>Expectation</td>
<td>$\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y) p_{X,Y}(x,y)$</td>
</tr>
<tr>
<td>Independence must have</td>
<td>$\forall x,y, p_{X,Y}(x,y) = p_X(x)p_Y(y)$</td>
</tr>
<tr>
<td>must have</td>
<td>$\Omega_{X,Y} = \Omega_X \times \Omega_Y$</td>
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</tbody>
</table>

(b) Law of Total Probability (r.v. version): If $X$ is a discrete random variable, then

$$ \mathbb{P}(A) = \sum_{x \in \Omega_X} \mathbb{P}(A | X = x)p_X(x) \quad \text{discrete } X $$

(c) Law of Total Expectation (Event Version): Let $X$ be a discrete random variable, and let events $A_1, \ldots, A_n$ partition the sample space. Then,

$$ \mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X | A_i] \mathbb{P}(A_i) $$

(d) Conditional Expectation: See table. Note that linearity of expectation still applies to conditional expectation: $\mathbb{E}[X + Y | A] = \mathbb{E}[X | A] + \mathbb{E}[Y | A]$

(e) Law of Total Expectation (RV Version): Suppose $X$ and $Y$ are random variables. Then,

$$ \mathbb{E}[X] = \sum_{y} \mathbb{E}[X | Y = y] p_Y(y) \quad \text{discrete version.} $$

(f) Conditional distributions (not covered in class)

<table>
<thead>
<tr>
<th>Discrete</th>
<th>Continuous</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conditional PMF/PDF</td>
<td>$p_{X</td>
</tr>
<tr>
<td>Conditional Expectation</td>
<td>$\mathbb{E}[X</td>
</tr>
</tbody>
</table>

(g) The following have not been covered as of 11/19:

- Law of Total Probability (continuous)

$$ \mathbb{P}(A) = \int_{x \in \Omega_X} \mathbb{P}(A | X = x) f_X(x) \, dx $$

- Law of total expectation (continuous)

$$ \mathbb{E}[X] = \int_{y \in \Omega_Y} \mathbb{E}[X | Y = y] f_Y(y) \, dy $$
2. Joint PMF’s
Suppose $X$ and $Y$ have the following joint PMF:

<table>
<thead>
<tr>
<th>$X/Y$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.2</td>
<td>0</td>
<td>0.1</td>
</tr>
<tr>
<td>1</td>
<td>0.3</td>
<td>0</td>
<td>0.4</td>
</tr>
</tbody>
</table>

(a) Identify the range of $X$ ($\Omega_X$), the range of $Y$ ($\Omega_Y$), and their joint range ($\Omega_{X,Y}$).

(b) Find the marginal PMF for $X$, $p_X(x)$ for $x \in \Omega_X$.

(c) Find the marginal PMF for $Y$, $p_Y(y)$ for $y \in \Omega_Y$.

(d) Are $X$ and $Y$ independent? Why or why not?

(e) Find $E[X^3Y]$.

3. Trinomial Distribution
A generalization of the Binomial model is when there is a sequence of $n$ independent trials, but with three outcomes, where $P$(outcome $i$) = $p_i$ for $i = 1, 2, 3$ and of course $p_1 + p_2 + p_3 = 1$. Let $X_i$ be the number of times outcome $i$ occurred for $i = 1, 2, 3$, where $X_1 + X_2 + X_3 = n$. Find the joint PMF $p_{X_1,X_2,X_3}(x_1, x_2, x_3)$ and specify its value for all $x_1, x_2, x_3 \in \mathbb{R}$.

4. Do You “Urn” to Learn More About Probability?
Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let $X_i = 1$ if the $i$-th ball selected is white and let it be equal to 0 otherwise. Give the joint probability mass function of

(a) $X_1, X_2$

(b) $X_1, X_2, X_3$

5. Successes
Consider a sequence of independent Bernoulli trials, each of which is a success with probability $p$. Let $X_1$ be the number of failures preceding the first success, and let $X_2$ be the number of failures between the first 2 successes. Find the joint pmf of $X_1$ and $X_2$. Write an expression for $E[\sqrt{X_1X_2}]$. You can leave your answer in the form of a sum.

6. Continuous joint density I
The joint probability density function of $X$ and $Y$ is given by

\[ f_{X,Y}(x, y) = \begin{cases} 
\frac{6}{7} \left( x^2 + \frac{xy}{2} \right) & 0 < x < 1, \ 0 < y < 2 \\
0 & \text{otherwise.} 
\end{cases} \]

(a) Verify that this is indeed a joint density function.

(b) Compute the marginal density function of $X$.

(c) Find $Pr(X > Y)$. (Uses the continuous law of total probability which we have not covered in class as of 11/19.)
(d) Find $P(Y > \frac{1}{2} | X < \frac{1}{2})$.
(e) Find $E(X)$.
(f) Find $E(Y)$.

7. Continuous joint density II
The joint density of $X$ and $Y$ is given by

$$f_{X,Y}(x, y) = \begin{cases} xe^{-(x+y)} & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

and the joint density of $W$ and $V$ is given by

$$f_{W,V}(w, v) = \begin{cases} 2 & 0 < w < v, 0 < v < 1 \\ 0 & \text{otherwise} \end{cases}$$

Are $X$ and $Y$ independent? Are $W$ and $V$ independent?

8. Trapped Miner
A miner is trapped in a mine containing 3 doors.
- $D_1$: The 1st door leads to a tunnel that will take him to safety after 3 hours.
- $D_2$: The 2nd door leads to a tunnel that returns him to the mine after 5 hours.
- $D_3$: The 3rd door leads to a tunnel that returns him to the mine after a number of hours that is Binomial with parameters $(12, \frac{1}{3})$.

At all times, he is equally likely to choose any one of the doors. What is the expected number of hours for this miner to reach safety?

9. Lemonade Stand
Suppose I run a lemonade stand, which costs me $100 a day to operate. I sell a drink of lemonade for $20. Every person who walks by my stand either buys a drink or doesn’t (no one buys more than one). If it is raining, $n_1$ people walk by my stand, and each buys a drink independently with probability $p_1$. If it isn’t raining, $n_2$ people walk by my stand, and each buys a drink independently with probability $p_2$. It rains each day with probability $p_3$, independently of every other day. Let $X$ be my profit over the next week. In terms of $n_1, n_2, p_1, p_2$ and $p_3$, what is $E[X]$?

10. Particle Emissions
Suppose we are measuring particle emissions, and the number of particles emitted follows a Poisson distribution with parameter $\lambda$, $X \sim \text{Poisson}(\lambda)$. Suppose our device to measure emissions is not always entirely accurate sometimes we fail to observe particles that actually emitted. So for each particle actually emitted, say we have probability $p$ of actually recording it, independently of other particles. Let $Y$ be the number of particles we observed. What distribution does $Y$ follow with what parameters, and what is $E[Y]$?

11. In between
(Covers ideas that have not been covered in class.) Suppose that $X_1$ and $X_2$ are discrete uniform random variables in $\{1, \ldots, 2n\}$ (i.e., $X_1$ and $X_2$ are equally likely to take any of the values $1, \ldots, 2n$) and let $Y = \min(X_1, X_2)$. What is the conditional pmf $p_{Y|X_1}(y \mid x_1)$ and conditional CDF $F_{Y|X_1}(y \mid x_1)$. What is
$E[Y \mid X_1 = x_1]$? (For the definitions of conditional pmf, conditional CDF, see the review at the top of this worksheet.)

12. 3 points on a line
(This problem uses the continuous law of total probability which has not yet be covered in class.) Three points $X_1, X_2, X_3$ are selected at random on a line $L$ (continuous independent uniform distributions). What is the probability that $X_2$ lies between $X_1$ and $X_3$?