CSE 312: Foundations of Computing II

Section 2: Intro Probability Solutions

1. Review of Main Concepts

(a) **Key Probability Definitions**

(a) **Sample Space:** The set of all possible outcomes of an experiment, denoted Ω or S

(b) **Event:** Some subset of the sample space, usually a capital letter such as $E \subseteq \Omega$

(c) **Union:** The union of two events E and F is denoted $E \cup F$

(d) **Intersection:** The intersection of two events E and F is denoted $E \cap F$ or EF

(e) **Mutually Exclusive:** Events E and F are mutually exclusive iff $E \cap F = \emptyset$

(f) **Complement:** The complement of an event E is denoted E^C or $\overline{E}$ or $\neg E$, and is equal to $\Omega \setminus E$

(g) **DeMorgan's Laws:** $(E \cup F)^C = E^C \cap F^C$ and $(E \cap F)^C = E^C \cup F^C$

(h) **Probability of an event E:** denoted $\mathbb{P}(E)$ or Pr(E) or $P(E)$

(i) **Partition:** Nonempty events $E_1, \ldots, E_n$ partition the sample space Ω iff

- $E_1, \ldots, E_n$ are exhaustive: $E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^n E_i = \Omega$, and
- $E_1, \ldots, E_n$ are pairwise mutually exclusive: $\forall i \neq j, E_i \cap E_j = \emptyset$
  - Note that for any event A (with $A \neq \emptyset, A \neq \Omega$): A and A^C partition Ω

(b) **Axioms of Probability and their Consequences**

(a) **Axiom 1: Non-negativity** For any event E, $\mathbb{P}(E) \geq 0$

(b) **Axiom 2: Normalization** $\mathbb{P}(\Omega) = 1$

(c) **Axiom 3: Countable Additivity** If E and F are mutually exclusive, then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$. Also, if $E_1, E_2, ...$ is a countable sequence of disjoint events, $\mathbb{P}(\bigcup_{k=1}^{\infty} E_i) = \sum_{k=1}^{\infty} \mathbb{P}(E_i)$.

(d) **Corollary 1: Complementation** $\mathbb{P}(E) + \mathbb{P}(E^C) = 1$

(e) **Corollary 2: Monotonicity** If $E \subseteq F$, $\mathbb{P}(E) \leq \mathbb{P}(F)$

(f) **Corollary 2: Inclusion-Exclusion** $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$

(c) **Equally Likely Outcomes:** If every outcome in a finite sample space Ω is equally likely, and E is an event, then $\mathbb{P}(E) = \dfrac{|E|}{|\Omega|}$.

- Make sure to be consistent when counting $|E|$ and $|\Omega|$. Either order matters in both, or order doesn't matter in both.

2. Probability by Simulation

In section, we'll work through this Edstem lesson on Probability by Simulation. This will be very helpful for the coding portion of Pset 2.

For problems 3-5 and 8, first answer the following two questions and then answer the question stated. (i) What is the sample space and how big is it? (ii) What is the probability of each outcome in the sample space? Unless otherwise specified, each outcome is equally likely.

3. Spades and Hearts

Given 3 different spades and 3 different hearts, shuffle them. Compute $\Pr(E)$, where E is the event that the suits of the shuffled cards are in alternating order.

Solution:

The sample space Ω is all re-orderings possible: there are $|\Omega| = 6!$ such. Now for E, order the spades and hearts independently, so there are $3!^2$ ways to do so. Finally choose whether you want hearts or spades first. All such orderings are equally likely, so $\Pr(E) = \frac{|E|}{|\Omega|} = \frac{2 \cdot 3!^2}{6!}$.

4. Trick or Treat

Suppose on Halloween, someone is too lazy to keep answering the door, and leaves a jar of exactly N total candies. You count that there are exactly K of them which are kit kats (and the rest are not). The sign says to please take exactly n candies. Each item is equally likely to be drawn. Let X be the number of kit kats we draw (out of n). What is $\Pr(X = k)$, that is, the probability we draw exactly k kit kats?

Solution:

Let E be the event that $X = k$, and the sample space be every way we can choose n candies out of N total.

$$\boxed{\Pr(X = k) = \frac{|E|}{|\Omega|} = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}}$$

In order to choose exactly k kit kats, we must choose k out of the K kit kats, and $n - k$ out of the $N - K$ other candies. The size of the sample space is just the number of ways to choose n candies out of N.

5. Staff Photo

Suppose we have 11 chairs (in a row) with 7 TA's, and Professors Karlin, Ruzzo, Rao, and Tompa to be seated. Suppose all seatings are equally likely. What is the probability that every professor has a TA to their immediate left and right?

Solution:

Imagine we permute all 7 TA's first – there are $7!$ ways to do this. Then, there are 6 spots between them that would result in a TA on both sides. We choose 4 of them for the Professors to sit – order matters since each Professor is distinct, so we multiply by $4!$. So the total ways is $7! \cdot \binom{6}{4} \cdot 4!$.

The sample space is the total number of ways to seat all 11 people: simply $11!$.

Since each seating is equally likely, the probability is then $\boxed{\frac{7! \cdot \binom{6}{4} \cdot 4!}{11!}}$

6. Weighted Die

Consider a weighted die such that

- $\Pr(1) = \Pr(2)$,
- $\Pr(3) = \Pr(4) = \Pr(5) = \Pr(6)$, and
- $\Pr(1) = 3\Pr(3)$.

What is the probability that the outcome is 3 or 4?

Solution:

By the second axiom of probability, the sum of probabilities for the sample space must equal 1. That is, $\sum_{i=1}^{6} \Pr(i) = 1$. Since $\Pr(1) = \Pr(2)$ and $\Pr(1) = 3\Pr(3)$, we have that: $1 = \Pr(1) + \Pr(2) + \Pr(3) + \Pr(4) + \Pr(5) + \Pr(6) = 3\Pr(3) + 3\Pr(3) + \Pr(3) + \Pr(3) + \Pr(3) + \Pr(3) = 10\Pr(3)$

Thus, solving algebraically, Pr$(3) = 0.1$, so Pr$(3) =$ Pr$(4) = 0.1$. Since rolling a 3 and 4 are disjoint events, then Pr$(3$ or $4) =$ Pr$(3) +$ Pr$(4) = 0.1 + 0.1 = 0.2$.

7. Fleas on Squares (Pigeonhole principle)

25 fleas sit on a 5×5 checkerboard, one per square. At the stroke of noon, all jump across an edge (not a corner) of their square to an adjacent square. At least two must end up in the same square. Why?

Solution:

There are two colors on a checkerboard; 13 are of one color, and 12 are of another. Each colored square is only surrounded by opposite colored squares on its edges. Therefore, the 13 fleas on the first color can only jump to a square of the second color — of which there are only 12 positions. So at least two fleas must land on the same square by the pigeonhole principle.

8. Congressional Tea Party

Twenty politicians are having a tea party, 6 Democrats and 14 Republicans.

(a) If they only give tea to 10 of the 20 people, what is the probability that they only give tea to Republicans?

Solution:

The sample space is the number of ways to give tea to people, so there are $\binom{20}{10}$ ways. The event is the ways to give tea to only Republicans, of which there are $\binom{14}{10}$ ways. So the probability is $\frac{\binom{14}{10}}{\binom{20}{10}}$.

(b) If they only give tea to 10 of the 20 people, what is the probability that they give tea to 8 Republicans and 2 Democrats?

Solution:

Similarly to the previous part, $\frac{\binom{14}{8}\binom{6}{2}}{\binom{20}{10}}$.