CSE 312 section 1

Made by Leiyi Zhang and Scott Ni

Review

- Sum Rule: no overlap in outcomes
- Product Rule: counting choices in stages
- Permutation: Ordering of ***N*** Distinct Objects: ***N!***
- k-Permutation: Ordering ***k*** of ***n*** Distinct Objects: P(n, k) $\frac{n!}{(n-k)!}$

- k-Combination: Choosing ***k*** of ***n*** Distinct Objects (order does not matter) : C(n ,k) $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Shout out to William Howard-Snyder and Mitchell Estberg for this slide!

Review

- Complementary counting: **total - opposite**
- Inclusion-Exclusion: +singles - doubles + triples - quads + ...
- Binomial Theorem: $\forall x, y \in \mathbb{R}, \forall n \in \mathbb{N}\colon\ (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$
- Multinomial coefficient: identical objects among distinct ones

$$\frac{n!}{n_1!\, n_2! \ldots n_k!} = \binom{n}{n_1, n_2 \ldots n_k}$$

- Pigeonhole principle: more candidate than possible places

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Question 2: “Seating”

How many ways are there to seat 10 people, consisting of 5 couples, in a row of 10 seats if . . .

(a) . . . all couples are to get adjacent seats?

(b) . . . anyone can sit anywhere, except that one couple insists on not sitting in adjacent seats?

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Answer: $5! * 2^5$ (By Product Rule)

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Let’s break it down!

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- The number of ways to arrange 10 people in 10 seats without any restrictions is: **10!**

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- The number of ways to arrange 10 people in 10 seats without any restrictions is: **10!**
- Then, we can treat that couple as a “ninth unit” added to the other 8 individuals, and then there are 2! ways to arrange that two people within their unit. Thus, the number of ways to rearrange people so that that couple will sit in adjacent seats is: **9! * 2!**

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- Finally, by using the method of “Complements”, we get $10! - 9! * 2 = 8 * 9!$

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(b) . . . anyone can sit anywhere, except that one couple insists on not sitting in adjacent seats?

Alternatively:

- Name the two people in the couple A and B. There are two cases:
  - A can sit on one of the ends, or not. If A sits on an end seat, A has 2 choices and B has 8 possible seats.
  - If A doesn't sit on the end, A has 8 choices and B only has 7.
- So there are a total of $2 \cdot 8 + 8 \cdot 7$ ways A and B can sit. Once they do, the other 8 people can sit in 8! ways since there are no other restrictions.

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    - If A doesn't sit on the end, A has 8 choices and B only has 7.
- So there are a total of 2·8+8·7 ways A and B can sit. Once they do, the other 8 people can sit in 8! ways since there are no other restrictions.

Answer: $(2 * 8 + 8 * 7)8! = 9 * 8 * 8! = 8 * 9!$

Question 3: “Weird Card Game”

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In how many ways can a pack of fifty-two cards be dealt to thirteen players, four to each, so that every player has one card of each suit?

- Deal one suit at a time. For each suit, there are 13! ways to distribute one card to each person.

  Answer: $13!^4$

Question 5: “A Team and a Captain”

Give a combinatorial proof of the following identity:

$$n\binom{n-1}{r-1} = \binom{n}{r}r$$

Hint: Consider two ways to choose a team of size r out of a set of size n and a captain of the team (who is also one of the team members).

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Definition 1.3.4.1: Combinatorial Proofs

To prove two quantities are equal, you can come up with a combinatorial situation, and show that both in fact count the same thing, and hence must be equal.

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*"first choose the captain (**n** ways to do that), and then choose the remaining members of the team from the remaining people"*

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How many ways are there to permute the 8 letters A, B, C, D, E, F, G, H so that A is not at the beginning and H is not at the end?

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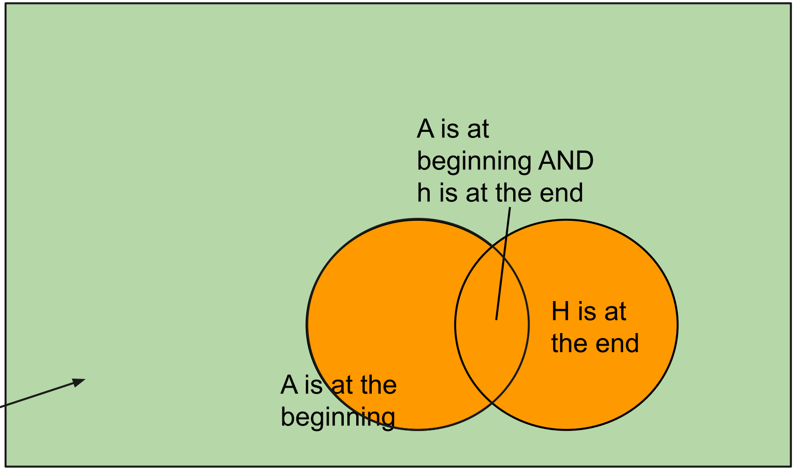
- **Principle of Inclusion and Exclusion!**

How many ways are there to permute the 8 letters A, B, C, D, E, F, G, H so that A is not at the beginning and H is not at the end?

A, B, C, D, E, F, G, H

- 8! ways to order them
- 7! for those with A at the beginning
- 7! for those with H at the end
- 6! for those with A at the beginning and H at the end

What we want to get (in green)



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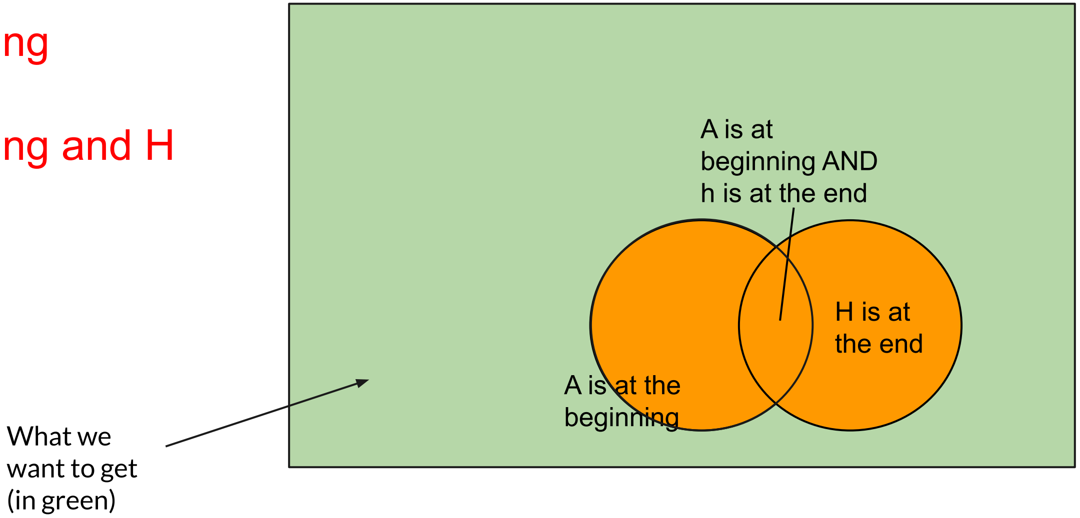
Answer: 8!-7!-7!+6!



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