Problem Set 2 (due Wednesday, October 14, 11:59pm)

Directions:

Answers: *For each problem, remember you must briefly explain/justify how you obtained your answer, as correct answers without an explanation will receive* ***no credit****. Moreover, in the event of an incorrect answer, we can still try to give you partial credit based on the explanation you provide. It is fine for your answers to include summations, products, factorials, exponentials, or combinations; you dont need to calculate those all out to get a single numeric answer, for instance* 26^7 *or* $26!/7!$ *or* $26 \cdot \binom{26}{7}$.

Your solutions need to be concise and clear. We will take off points for lack of clarity or for excess verbosity. Please see section worksheet solutions (posted on the course website) to gauge the level of detail we are expecting.

Please clearly indicate your final answer, in such a way as to distinguish it from the rest of your explanation.

Groups: *This pset may be done with a* ***single partner****. In this case, only one person will submit on Gradescope and add their partner as a collaborator. Contrary to what we originally said in the syllabus, we've decided to make this apply to both the written and coding portion.* ***However,*** *you will be doing some future coding on your own so make sure both partners are engaged and participating.*

Individuals and pairs are still encouraged to discuss problem-solving strategies with other classmates as well as the course staff, but each pair must write up their own solutions.

Submission: *You must upload a* ***pdf*** *of your written solutions to Gradescope under "PSet 2 [Written]". Problem 7 has a coding and written portion, so under "Pset 2 [Coding]" you will be uploading a pdf of your written solutions (including the plot we ask you to make) and a .py file called cse312_pset2_pingpong.py. If you do the extra credit, that will be submitted seperately under "Pset2 [Extra]". (Instructions as to how to upload your solutions to gradescope are on the course web page.) The use of latex is highly recommended.*

Note that if you want to hand-write your solutions, you'll need to scan them. We will take off points for hand-written solutions that are difficult to read due to poor handwriting and neatness.)

Please cite any collaboration at the top of your submission (beyond your group members, which should already be listed).

1. Combinatorial Identities (16 points)

Prove each of the following identities using a *combinatorial argument*; an algebraic solution will be marked substantially incorrect. (Note that $\binom{a}{b}$ is 0 if $b > a$.)

(a) [8 Points] $\displaystyle\sum_{k=0}^{\infty} \binom{m}{k}\binom{n}{k} = \binom{m+n}{n}$.

Hint: Start with the right hand side and imagine you are choosing a team of n people from a group of people consisting of m Americans and n Canadians.

(b) [8 Points] $\displaystyle\sum_{k=0}^{\infty} \binom{n}{k}\binom{k}{m} = \binom{n}{m}2^{n-m}$.

2. Rotating the table (10 points)

At a dinner party, all of the n people present are to be seated at a circular table. Suppose there is a nametag at each place at the table and suppose that nobody sits down in their correct place. Use the pigeon-hole principle to show that it is possible to rotate the table so that at least two people are sitting in the correct place. Be sure to specify precisely what the pigeons are, precisely what the pigeonholes are, and precisely what the mapping of pigeons to pigeonholes is.

3. Stuff into stuff (12 points)

(a) [4 Points] We have 10 people and 30 rooms. How many different ways are there to assign the (distinguishable) people to the (distinguishable) rooms? (Any number of people can go into any of the 30 rooms.)

(b) [4 Points] We have 20 identical (indistinguishable) apples. How many different ways are there to place the apples into 30 (distinguishable) boxes? (Any number of apples can go into any of the boxes.)

(c) [4 Points] We have 30 identical (indistinguishable) apples. How many different ways are there to place the apples into 8 (distinguishable) boxes, if each box is required to have at least two apples in it.?

4. Sample Spaces and Probabilities (18 points)

For each of the following scenarios first describe the sample space and indicate how big it is (i.e., what its cardinality is) and then answer the question.

(a) [3 Points] You flip a fair coin 50 times. What is the probability of exactly 20 heads?

(b) [3 Points] You roll 2 fair 6-sided dice, one red and one blue. What is the probability that the sum of the two values showing is 4?

(c) [3 Points] You are given a random 5 card poker hand (selected from a single deck). What is the probability you have a full-house (3 cards of one rank and 2 cards of another rank)?

(d) [3 Points] 20 labeled balls are placed into 10 labeled bins (with each placement equally likely). What is the probability that bin 1 contains exactly 3 balls?

(e) [3 Points] There are 30 psychiatrists and 24 psychologists attending a certain conference. Three of these 54 people are randomly chosen to take part in a panel discussion. What is the probability that at least one psychologist is chosen? What is the probability that exactly three psychologists are chosen?

(f) [3 Points] You buy ten cupcakes choosing from 3 different types (chocolate, vanilla and caramel). Cupcakes of the same type are indistinguishable. What is the probability that you have at least one of each type?

5. Miscounting (9 points)

Consider the question: what is the probability of getting a **7-card** poker hand (order doesn't matter) that contains at least two 3-of-a-kind (3-of-a-kind means three cards of the same rank). For example, this would be a valid hand: ace of hearts, ace of diamonds, ace of spaces, 7 of clubs, 7 of spades, 7 of hearts and queen of clubs. (Note that a hand consisting of all 4 aces and three of the 7s is also valid.)

Here is how we might compute this:

> Each of the $\binom{52}{7}$ hands is equally likely. Let E be the event that the hand selected contains at least two 3-of-a-kinds. Then
>
> $$\Pr(E) = \frac{|E|}{\binom{52}{7}}$$

To compute $|E|$, apply the product rule. First pick two ranks that have a 3-of-a-kind (e.g. ace and 7 in the example above). For the lower rank of these, pick the suits of the three cards. Then for the higher rank of these, pick the suits of the three cards. Then out of the remaining $52 - 6 = 46$ cards, pick one. Therefore

$$|E| = \binom{13}{2} \cdot \binom{4}{3} \cdot \binom{4}{3} \cdot \binom{46}{1} \qquad \text{and hence} \qquad \Pr(E) = \frac{\binom{13}{2} \cdot 4^2 \cdot 46}{\binom{52}{7}}.$$

Explain what is wrong with this solution. If there is over-counting in $|E|$, characterize all hands that are counted more than once, and how many times each such hand is counted. If there is under-counting in $|E|$, explain which hands are not counted.
Also, give the correct answer for $\Pr(E)$.

6. Random Questions (15 points)

(a) [5 Points] What is the probability that the digit 1 doesn't appear among n digits where each digit is one of (0-9) and all sequences are equally likely?

(b) [5 Points] Suppose you randomly permute the numbers $1, 2, \ldots, n$, (where $n > 500$). That is, you select a permutation uniformly at random. What is the probability that the number 3 ends up in the 130-th position in the resulting permutation? (For example, in the permutation $1, 3, 2, 5, 4$ of the numbers $1 \ldots 5$, the number 2 is in the 3rd position in the permutation and the number 4 is in the 5th position.)

(c) [5 Points] A fair coin is flipped n times (each outcome in $\{H, T\}^n$ is equally likely). What is the probability that all heads occur at the end of the sequence? (The case that there are no heads is a special case of having all heads at the end of the sequence, i.e. 0 heads.)

7. Ping Pong [coding + written] (20 points)

We'll finally answer the long-awaited question: what's the probability you win a ping pong game up to n points, when your probability of winning each point is p (and your friend wins the point with probability $1-p$)? Assume you have to win by (at least) 2; for example, if $n = 21$ and the score is $21 - 20$, the game isn't over yet.
Write your code for the following parts in the provided file: cse312_pset2_pingpong.py.

(a) [5 Points] Implement the function part_a.

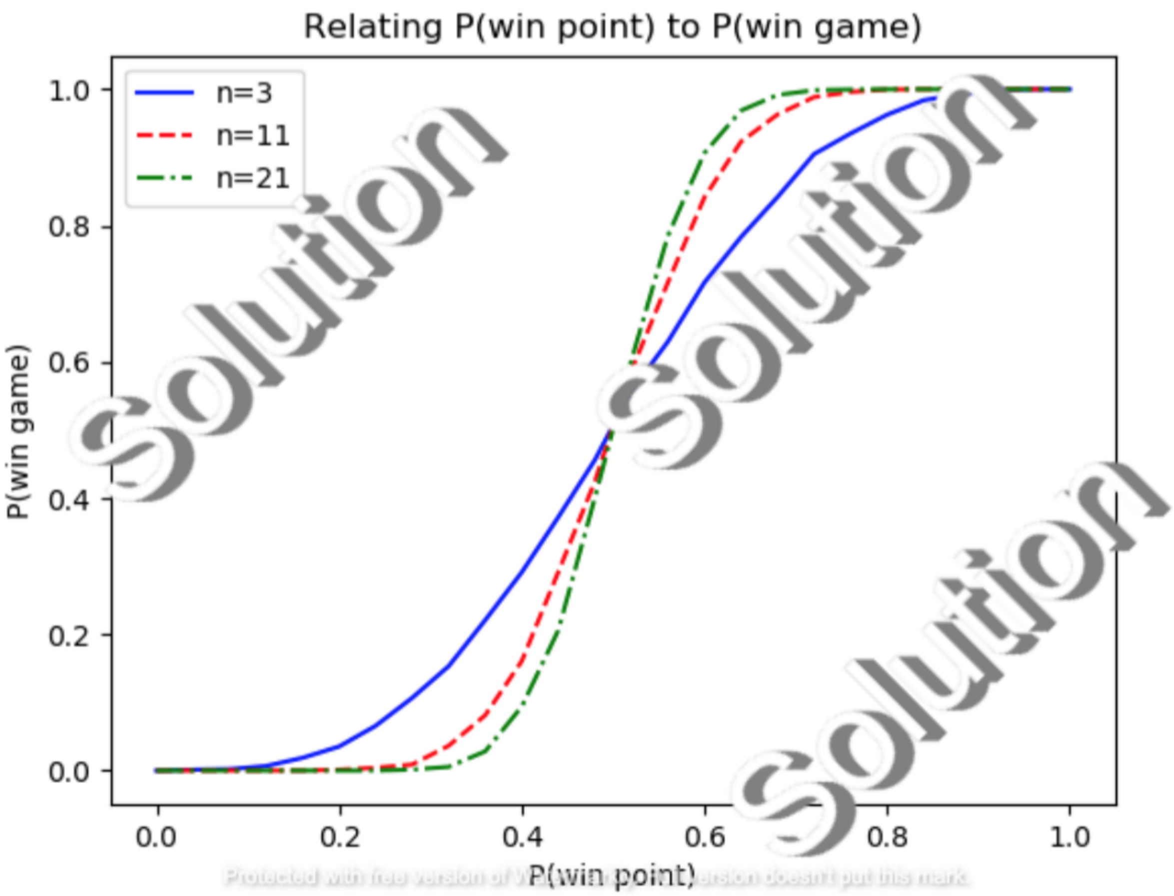
(b) [15 Points] Implement the function part_b. This function will NOT be autograded but you will still submit it; you should use the space here to generate the plot asked of you below.

i. Generate the plot below in Python (without the watermarks). Details on how to construct it are in the starter code. Attach your plot in your written submission for this part.

ii. Write AT MOST 2-3 sentences identifying the interesting pattern you notice when n gets larger (regarding the steepness of the curve), and explain why it makes sense.

iii. Each curve you make for different values of n always (approximately) passes through 3 points. Give the three points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, and explain why intuitively this happens in AT MOST 2-3 sentences.

Figure 1: Your plot should look something like this.



8. Extra Credit (5 points)

Consider the ping pong scenario from the previous problem. For $n = 21$, compute the **exact** probability of winning a game, as a function of the probability of winning a single point p. Your answer may include summations and binomial coefficients. Then, evaluate your answer when $p = 0.3$ and give your answer to 6 decimal places. (Hint: Consider two cases; one where your final score was 21, and one where you had to play until you won by 2.)