Review: Main Theorems and Concepts

Binomial Theorem: \( \forall x, y \in \mathbb{R}, \forall n \in \mathbb{N}: (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \)

Principle of Inclusion-Exclusion (PIE): 2 events: \(|A \cup B| = |A| + |B| - |A \cap B|\)
3 events: \(|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|\)
In general: \( + \text{singles} - \text{doubles} + \text{triples} - \text{quads} + \ldots \)

Pigeonhole Principle: If there are \(n\) pigeons with \(k\) holes and \(n > k\), then at least one hole contains at least 2 (or to be precise, \(\lceil \frac{n}{k} \rceil\)) pigeons.

Complementary Counting (Complementing): If asked to find the number of ways to do X, you can: find the total number of ways and then subtract the number of ways to not do X.

Sample Space: The set of all possible outcomes of an experiment, denoted \(\Omega\) or \(S\)
Event: Some subset of the sample space, usually a capital letter such as \(E \subseteq \Omega\)
Union: The union of two events \(E\) and \(F\) is denoted \(E \cup F\)
Intersection: The intersection of two events \(E\) and \(F\) is denoted \(E \cap F\) or \(EF\)
Mutually Exclusive: Events \(E\) and \(F\) are mutually exclusive iff \(E \cap F = \emptyset\)
Complement: The complement of an event \(E\) is denoted \(E^C\) or \(\overline{E}\) or \(\neg E\), and is equal to \(\Omega \setminus E\)
DeMorgan’s Laws: \((E \cup F)^C = E^C \cap F^C\) and \((E \cap F)^C = E^C \cup F^C\)
Probability of an event \(E\): denoted \(P(E)\) or \(Pr(E)\) or \(P(E)\)
Partition: Nonempty events \(E_1, \ldots, E_n\) partition the sample space \(\Omega\) iff
- \(E_1, \ldots, E_n\) are exhaustive: \(E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega\), and
- \(E_1, \ldots, E_n\) are pairwise mutually exclusive: \(\forall i \neq j, E_i \cap E_j = \emptyset\)
  - Note that for any event \(A\) (with \(A \neq \emptyset, A \neq \Omega\)): \(A\) and \(A^C\) partition \(\Omega\)

Axioms of Probability and their Consequences

1. (Non-negativity) For any event \(E\), \(P(E) \geq 0\)
2. (Normalization) \(P(\Omega) = 1\)
3. (Additivity) If \(E\) and \(F\) are mutually exclusive, then \(P(E \cup F) = P(E) + P(F)\)

Corollaries of these axioms:
• \( \mathbb{P}(E) + \mathbb{P}(E^C) = 1 \)
• If \( E \subseteq F \), \( \mathbb{P}(E) \leq \mathbb{P}(F) \)
• \( \mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F) \)

**Equally Likely Outcomes:** If every outcome in a finite sample space \( \Omega \) is equally likely, and \( E \) is an event, then \( \mathbb{P}(E) = \frac{|E|}{|\Omega|} \).

• Make sure to be consistent when counting \( |E| \) and \( |\Omega| \). Either order matters in both, or order doesn’t matter in both.

**Conditional Probability:** \( \mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \)

**Exercises**

1. Give a **combinatorial** proof that \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \). Do not use the binomial theorem. (Hint: you can count the number of subsets of \([n] = \{1, 2, \ldots, n\}\). Note: A combinatorial proof is one in which you explain how to count something in two different ways – then those formulae must be equivalent if they both indeed count the same thing.

   Fix a subset of \([n] \) of size \( k \). There are \( \binom{n}{k} \) such subsets because we choose any \( k \) elements out of the \( n \), with order not mattering since these are sets. Subsets can be of size \( k = 0, 1, \ldots, n \). So the total number of subsets of \([n] \) is \( \sum_{k=0}^{n} \binom{n}{k} \). On the other hand, each element of \([n] \) is either in a subset or not. So there are 2 possibilities for the first element (in or out), 2 for the second, etc. By the product rule, then, there are \( 2^n \) subsets of \([n] \). Therefore, \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \). Note that this agrees with the binomial theorem using \( x = y = 1 \).

2. How many ways are there to choose three initials (upper case letters) such that two are the same or all three are the same?

   Complementary counting: Count the total \( 26^3 \) and subtract the number with all distinct initials \( 26 \cdot 25 \cdot 24 \) to get \( 26^3 - 26 \cdot 25 \cdot 24 \).

3. Suppose there are \( N \) items in a bag, with \( K \) of them marked as successes in total (and the rest are marked as failures). We draw \( n \) of them, without replacement. Each item is equally likely to be drawn. Let \( X \) be the number of successes we draw (out of \( n \)). What is \( \mathbb{P}(X = k) \), that is, the probability we draw exactly \( k \) successes?

\[
\mathbb{P}(X = k) = \binom{K}{k} \binom{N-K}{n-k} \binom{N}{n}
\]
We choose \( k \) out of the \( K \) successes, and \( n - k \) out of the \( N - K \) failures. The denominator is the total number of ways to choose \( n \) objects out of \( N \).

4. Suppose we have 12 chairs (in a row) with 9 TA’s, and Professors Ruzzo, Karlin, and Tompa to be seated. Suppose all seatings are equally likely. What is the probability that every professor has a TA to his/her immediate left and right?

Imagine we permute all 9 TA’s first – there are \( 9! \) ways to do this. Then, there are 8 spots between them, in which we pick 3 for the Professors to sit – order matters since each Professor is distinct. So the total ways is \( P(8, 3) \cdot 9! \).

The total number of ways to seat all 12 of us is simply \( 12! \).

The probability is then \( \frac{P(8, 3) \cdot 9!}{12!} \).

5. Suppose Joe is a \( k \)-legged robot, who wears a sock and a shoe on each leg. Suppose he puts on \( k \) socks and \( k \) shoes in some order, each equally likely. Each action is specified by saying whether he puts on a sock or a shoe, and saying which leg he puts it on. In how many ways can he put on his socks and shoes in a valid order? We say an ordering is valid if, for every leg, the sock gets put on before the shoe. Assume all socks are indistinguishable from each other, and all shoes are indistinguishable from each other.

First, note there are \( 2k \) objects – \( k \) shoes and \( k \) socks. Suppose we describe a sequence of actions,

\[ Sock_1, Shoe_1, Sock_2, Shoe_2, \ldots, Sock_k, Shoe_k. \]

This particular example means we first put a sock on leg 1, then a shoe on leg 1, then a sock on leg 2, etc. There are \( (2k)! \) ways to order these actions. However, for each leg, there is only one valid ordering: the sock must come before the shoe. So we divide by \( 2^k \) and the total number of ways is \( \frac{(2k)!}{2^k} \).

Alternatively, \( P(\text{valid ordering}) = \frac{\text{|valid orderings|}}{\text{|orderings|}} \), so \( \text{|valid orderings|} = P(\text{valid ordering}) \cdot \text{|orderings|} \). We can compute \( P(\text{valid ordering}) = (1/2)^k \). Notice for any sequence of actions with each equally likely, the probability that the sock came before the shoe on a particular leg is \( 1/2 \), so the probability this happened for each leg is \( (1/2)^k \). Then \( \text{|orderings|} = (2k)! \) because there are \( 2k \) actions that we can permute, all distinct. Multiplication gives the same answer as above.

6. Find the number of ways to rearrange the word “INGREDIENT”, such that no two identical letters are adjacent to each other. For example, “INGREEDINT” is invalid because the two E’s are adjacent. Repeat the question for the letters “AAAAAABBB”.

We use inclusion-exclusion. Let \( \Omega \) be the set of all anagrams (permutations) of “INGREDIENT”, and \( A_I \) be the set of all anagrams with two consecutive I’s. Define \( A_E \) and \( A_N \) similarly. \( A_I \cup A_E \cup A_N \) clearly are the set of anagrams we don’t want. So we use complementing to count the size of \( \Omega \setminus (A_I \cup \ldots) \).
\( A_E \cup A_N \). By inclusion exclusion, \(|A_I \cup A_E \cup A_N| = \text{singles-doubles+triples}, \) and by complementing, 
\(|\Omega \setminus (A_I \cup A_E \cup A_N)| = |\Omega| - |A_I \cup A_E \cup A_N|.

First, \(|\Omega| = \frac{10!}{2!2!2!} \) because there are 2 of each of I,E,N’s (multinomial coefficient). Clearly, the size of \( A_I \) is the same as \( A_E \) and \( A_N \). So \(|A_I| = \frac{9!}{2!} \) because we treat the two adjacent I’s as one entity. We also need \(|A_I \cap A_E| = \frac{8!}{2!} \) because we treat the two adjacent I’s as one entity and the two adjacent E’s as one entity (same for all doubles). Finally, \(|A_I \cap A_E \cap A_N| = 7! \) since we treat each pair of adjacent I’s, E’s, and N’s as one entity.

Putting this together gives
\[
\frac{10!}{2!2!2!} - \left( \binom{3}{1} \cdot \frac{9!}{2!} - \binom{3}{2} \cdot \frac{8!}{2!} + \binom{3}{3} \cdot 7! \right)
\]

For the second question, note that no A’s and no B’s can be adjacent. So let us put the B’s down first:

By the pigeonhole principle, two A’s must go in the same slot, but then they would be adjacent, so there are \( \text{no ways} \).

7. Given 3 different spades and 3 different hearts, shuffle them. Compute \( P(E) \), where \( E \) is the event that the suits of the shuffled cards are in alternating order. What is your sample space?

The sample space is the set of all possible orderings of the 6 cards.

\[
P(E) = \frac{6 \cdot 3 \cdot 2 \cdot 1 \cdot 1}{6!} = \frac{1}{10}
\]

8. Suppose you pick two cards from a well-shuffled Schnapsen deck. What is the probability that they are both queens?

\[
\frac{4 \cdot 3}{20 \cdot 19} = \frac{3}{95} \approx 0.0316
\]

Alternatively,

\[
\binom{4}{2} \binom{20}{2}
\]

Notice that, in the first solution, we are consistent in numerator and denominator that order matters. In the second solution, we are consistent in numerator and denominator that order does not matter.

9. At a card party, someone brings out a deck of bridge cards (4 suits with 13 cards in each). \( N \) people each pick 2 cards from the deck and hold onto them. What is the minimum value of \( N \) that guarantees at least 2 people have the same combination of suits?
There are \( \binom{4}{2} \) combinations of 2 different suits, plus 4 possibilities of having 2 cards of the same suit. With \( N = 11 \) you can apply the pigeonhole principle.

10. Suppose you deal 13 cards from a well-shuffled bridge deck (4 suits with 13 cards in each). What is the probability that the distribution of suits is 4, 4, 3, 2? (That is, you have 4 cards of one suit, 4 cards of another suit, 3 cards of another suit, and 2 cards of the last suit.)

\[
\frac{\binom{13}{4} \binom{13}{4} \binom{13}{3} \binom{13}{2} \cdot 4!}{\binom{52}{13} \cdot 2!}
\]

The factor of 4! in the numerator takes care of the number of ways to assign suits to the number of cards, and the factor of 2! in the denominator takes care of the fact that two suits have the same number (4) of cards and so are overcounted.

11. Novice poker players are often confused about which player wins if one holds a flush and one holds a straight. For draw poker (see quiz section #1 worksheet, exercise #25):

(a) Compute the probability of being dealt a flush.

\[
\frac{\binom{4}{1} \binom{13}{5}}{\binom{52}{5}} \approx 0.00198
\]

(b) Compute the probability of being dealt a straight.

\[
\frac{10 \cdot 4^5}{\binom{52}{5}} \approx 0.00394
\]

(c) Which of these hands should win, given your answers to (a) and (b)?

A flush should beat a straight, since it is rarer.

12. This is another poker exercise. Find the minimum number of cards to be dealt to you from a standard 52-card deck to guarantee that you have some 5 cards among them that form . . .

(a) one pair? (This occurs when the cards have ranks a, a, b, c, d, where a, b, c, and d are all distinct. The suits do not matter.)

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(b) two pairs? (This occurs when the cards have ranks a, a, b, b, c, where a, b, and c are all distinct. The suits do not matter.)

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(c) a full house? (This occurs when the cards have ranks a, a, a, b, b, where a and b are distinct. The suits do not matter.)

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(d) a straight? (A hand is said to form a straight if the ranks of all 5 cards form an incrementing sequence. The suits do not matter. The lowest straight is A, 2, 3, 4, 5 and the highest straight is 10, J, Q, K, A.)

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(e) a flush? (A hand is said to form a flush if all 5 cards are from the same suit.)

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(f) a straight flush (5 cards of the same suit that form a straight)?

45

13. In Schnapsen, suppose that ♠J is the face-up trump and you are dealt 5 nontrump cards. Let $E$ be the event that the top 4 cards in the stock are all trumps. Let the sample space be all possible orderings of all the cards in the stock. Compute $P(E)$. (Notice that your solution suggests a different and simpler sample space.)

$$P(E) = \frac{4 \cdot 3 \cdot 2 \cdot 1 \cdot 10!/5!}{14!/5!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{14 \cdot 13 \cdot 12 \cdot 11}$$

The final answer suggests the simpler sample space of all possible orderings of just the top 4 cards in the stock.

14. Suppose you are taking a multiple-choice test that has $c$ answer choices for each question. In answering a question on this test, the probability that you know the correct answer is $p$. If you don’t know the answer, you choose one at random, with each choice equally probable. What is the probability that you knew the correct answer to a question, given that you answered it correctly?

$$P = \frac{p}{p + (1 - p)^\frac{1}{c}}$$

15. An urn contains 3 black balls and 4 white balls.
(a) Suppose 3 balls are drawn from the urn without replacement. What is the probability that all 3 are white? Try computing this in the sample space where the order of the 3 draws does not matter, and then in the sample space where the order does matter.

When order does not matter:

\[
\binom{4}{3} \cdot \binom{7}{3} = \frac{4 \cdot 3!}{7 \cdot 6 \cdot 5} = \frac{4}{35} \approx 0.114
\]

When order does matter:

\[
\frac{4!}{(4 - 3)!} \cdot \frac{7!}{(7 - 3)!} = \frac{4 \cdot 3 \cdot 2}{7 \cdot 6 \cdot 5} = \frac{4}{35} \approx 0.114
\]

(b) Suppose 3 balls are drawn from the urn with replacement. What is the probability that all 3 are white? Describe the sample space precisely.

The sample space consists of all ways of drawing 3 balls with replacement, where the order of the 3 draws matters. The probability is

\[
\frac{4^3}{7^3} = \left(\frac{4}{7}\right)^3 \approx 0.187
\]

16. At a dinner party, the \(n\) people present are to be seated uniformly spaced around a circular table. Suppose there is a nametag at each place at the table and suppose that nobody sits down at the correct place. Show that it is possible to rotate the table so that at least two people are sitting in the correct place.

For each of the \(n\) people, consider the offset (in a clockwise direction) of that person from their nametag. Since no one is sitting at the correct place in the table’s current position, the offsets are each in the set \(\{1, 2, \ldots, n - 1\}\). By the pigeonhole principle, two people must have the same offset. Rotate the table by that offset.

17. (a) Two parents only have 3 bedrooms for their 13 children. If each child is assigned to a bedroom, one of the bedrooms must have at least \(c\) children. What is the maximum value of \(c\) that makes this statement true? Prove it.

\(c = 5\). Prove that \(c > 4\) by contradiction.

(b) (Strong Pigeonhole Principle) More generally, what can you say about \(n\) children in \(k\) bedrooms? Find a general formula for the maximum value of \(c\) that guarantees one of the bedrooms must have at least \(c\) children.

\(c = \lceil n/k \rceil\). Note that the ordinary Pigeonhole Principle is the special case when \(k = n - 1\).
18. Twenty politicians are having a tea party, 6 Democrats and 14 Republicans.

(a) If they only give tea to 10 of the 20 people, what is the probability that they only give tea to Republicans?

\[ \frac{\binom{14}{10}}{\binom{20}{10}} \]

(b) If they only give tea to 10 of the 20 people, what is the probability that they give tea to 9 Republicans and 1 Democrat?

\[ \binom{14}{9} \binom{6}{1} \binom{20}{10} \]

19. A couple has 2 children. What is the probability that both are girls, given that the older one is a girl?

\[ \frac{1}{2} \], because the genders of the two children are independent.

20. What is the probability that at least one of a pair of fair dice comes up 5, given that the sum of the dice is 8?

Let \( D_1 \) be the outcome of the first die roll and \( D_2 \) be the outcome of the second die roll.

\[
\Pr(D_1 = 5 \cup D_2 = 5 \mid D_1 + D_2 = 8) = \frac{\Pr(D_1 + D_2 = 8 \cap (D_1 = 5 \cup D_2 = 5))}{\Pr(D_1 + D_2 = 8)}
\]

\[ = \frac{\Pr((D_1 = 5 \cap D_2 = 3) \cup (D_1 = 3 \cap D_2 = 5))}{\Pr(D_1 + D_2 = 8)}
\]

\[ = \frac{1/36 + 1/36}{5/36}
\]

\[ = \frac{2}{5} \]

Alternatively, you can restrict the sample space to the 5 equally likely outcomes in which \( D_1 + D_2 = 8 \) (that is, \((D_1, D_2) \in \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}) and observe that 2 of them include a 5.

21. A plane has 100 seats and 100 passengers. The first person to get on the plane lost his ticket and doesn’t know his assigned seat, so he picks a random seat to sit in, with each seat equally probable. Every remaining person knows their seat, so if it is available they sit in it, and if it is unavailable they
pick a random remaining seat, with each unoccupied seat equally probable. What is the probability the last person to get on gets to sit in his own seat?

We will prove that only the first and last person’s seat could possibly be available by the time the last person boards. Suppose for contradiction that the i-th person’s seat, where $2 \leq i \leq 99$, was available at the end. Then that i-th person’s seat was available the whole time, and the i-th person would have sat there after boarding, which is a contradiction. So only the first or last person’s seat could possibly be left when the last person boards.

One of those two seats must be occupied when the last person boards. Consider the person who took that seat, say passenger j, where $1 \leq j \leq 99$. When j boards, both the first and last person’s seats are still available. Passenger j is equally likely to take either of those two seats. Therefore, the probability that the last person’s seat is available when the last person boards is $\frac{1}{2}$.

22. (The “Monty Hall” puzzle) Suppose you’re on a game show, and you’re given the choice of three doors. Behind one door is a car, behind the others, goats. You pick a door, say number 1, and the host, who knows what’s behind the doors, opens another door, say number 3, which has a goat. He says to you, “Do you want to pick door number 2?” Is it to your advantage to switch your choice of doors?

It’s always better to switch doors. Think of this problem with 100 doors instead of 3 doors. When you pick one of them, the game show host opens 98 other doors all revealing goats, so you’re left with the door you picked and the remaining door. What would you pick in this scenario? You had probability $\frac{1}{100}$ of being right on your first pick, and probability $\frac{99}{100}$ that the car was behind some other door. Now that the 98 doors have been revealed, the probability is $\frac{99}{100}$ that the car is behind the remaining closed door that you didn’t pick.

23. (Challenge problem) n people at a reception give their hats to a hat-check person. When they leave, the hat-check person gives each of them a hat chosen at random. What is the probability that no one gets their own hat back?