Chebyshev's Inequality: Suppose we also know $Var(X)$.

Theorem: If Y is a r.v. with $E[Y]=\mu$, then for any $\alpha>0$, $P(|Y-\mu|\geq\alpha)\leq Var(Y)/\alpha^2$. Equivalently, if $\sigma=\sqrt{Var(Y)}$, then for any $t>0$,
$$P(|Y-\mu|\geq t\sigma)\leq 1/t^2.$$

Proof: Let $X=(Y-\mu)^2$. X is nonnegative, so
$$P(|Y-\mu|\geq\alpha)=P(X\geq\alpha^2)\leq\frac{E[X]}{\alpha^2}$$
$$=\frac{E[(Y-\mu)^2]}{\alpha^2}=\frac{Var(Y)}{\alpha^2}$$

Ex: Y is daily business cost, $E[Y]=1500$, and $Var(Y)=\sigma^2=(200)^2$.
$$P(Y\geq 2500)=P(Y-1500\geq 1000)=P(Y-\mu\geq 1000)$$
$$\leq P(|Y-\mu|\geq 1000)\leq\frac{200^2}{1000^2}=\left(\frac{1}{5}\right)^2=\frac{1}{25}$$

Cantelli's Inequality (one-sided Chebyshev)

If $\alpha>0$, $P(Y-\mu\geq\alpha)\leq\dfrac{Var(Y)}{Var(Y)+\alpha^2}$.

Ex: $\frac{1}{26}$: $P(Y\geq 2500)=P(Y-1500\geq 1000)\leq\dfrac{200^2}{200^2+1000^2}$
$$=\frac{1}{1+5^2}=\frac{1}{26}$$

Chernoff Bound

Theorem: Suppose $X \sim Bin(n,p)$ and $\mu = E[X]$. For any $0 < \delta < 1$,

$$P(X \geq (1+\delta)\mu) \leq e^{-\frac{1}{3}\delta^2 \mu} \text{ and}$$

$$P(X \leq (1-\delta)\mu) \leq e^{-\frac{1}{2}\delta^2 \mu}$$

Law of large numbers

Consider i.i.d. r.v.'s $X_1, X_2, \cdots$, where $E[X_i] = \mu < \infty$ and $Var(X_i) = \sigma^2 < \infty$.

Define sample mean $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

$$E[\overline{X}_n] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n}\cdot n \cdot \mu = \mu.$$

$$Var(\overline{X}_n) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n^2}\cdot n\sigma^2 = \frac{\sigma^2}{n}$$

As n increase, $\overline{X}_n$ is more likely to be ~~ed~~ close to μ.

Theorem (Weak Law of Large Numbers): For any $\varepsilon > 0$, as $n \to \infty$, $P(|\overline{X}_n - \mu| > \varepsilon) \to 0$.

Proof: By Chebyshev's Inequality,

$$P(|\overline{X}_n - \mu| > \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2} \to 0 \text{ as } n \to \infty.$$

Probabilistic Algorithms

Background:

Given an n-digit number x, can it be factored into its prime factors efficiently?

In time some small polynomial in n. E.g., $O(n^2)$

Big open problem today.

Simpler question: determine if x is prime or composite?

Solovay & Strassen 1977: a probabilistic polynomial time algorithm to determine if x is prime or composite

Quicksort (Hoare 1959)

To sort $a_1, a_2, \ldots, a_n$: If $n > 1$

1. Choose $p \in \{1, 2, \ldots, n\}$ uniformly and randomly.
   $p \sim \mathrm{Unif}(1, n)$.
2. Let $L = \{a_i \mid a_i < a_p\}$
   $E = \{a_i \mid a_i = a_p\}$
   $G = \{a_i \mid a_i > a_p\}$.
3. Recursively sort and output L
   Output E
   Recursively sort and output G.

The expected time used by quicksort is $O(n \log n)$.