Tail Bounds:
Bound the probability of being far from the mean.

Ex: Expected business cost per day is $1500. What is the probability that your cost on a particular day is $6000.

Markov's Inequality

Theorem: If $X$ is a nonnegative r.v., then for any $\alpha > 0$, $P(X \geq \alpha) \leq \frac{E[X]}{\alpha}$.

Equivalently, $P(X \geq kE[X]) \leq \frac{1}{k^2}$, for any $k > 0$.

Ex: If $X$ is your cost on the particular day, $P(X \geq 6000) \leq 1500/6000 = \frac{1}{4}$.
If $X = 6000$ with prob $\frac{1}{4}$, 0 with prob $\frac{3}{4}$, this is best bound possible.

Proof of Theorem:

\[ E[X] = \sum_{\alpha} \alpha P_{X}(\alpha) = \sum_{\alpha < \alpha} \alpha P_{X}(\alpha) + \sum_{\alpha \geq \alpha} \alpha P_{X}(\alpha) \]
\[ \geq 0 + \sum_{\alpha \geq \alpha} \alpha P_{X}(\alpha) = \alpha \sum_{\alpha \geq \alpha} P_{X}(\alpha) = \alpha P(X \geq \alpha) \]
So $P(X \geq \alpha) \leq \frac{E[X]}{\alpha}$.
Chebyshev's Inequality: Suppose we also know $\text{Var}(X)$

**Theorem:** If $Y$ is a r.v. with $E[Y] = \mu$, then for any $\alpha > 0$, 
$$P(|Y - \mu| \geq \alpha) \leq \frac{\text{Var}(Y)}{\alpha^2}.$$ 
Equivalently, if $\sigma = \sqrt{\text{Var}(Y)}$, then for any $t > 0$, 
$$P(|Y - \mu| \geq t\sigma) \leq \frac{1}{t^2}.$$

**Proof:** Let $X = (Y - \mu)^2$. $X$ is nonnegative, so 
$$P(|Y - \mu| \geq \alpha) = P(X \geq \alpha^2) \leq \frac{E[X]}{\alpha^2} \leq \frac{E[(Y - \mu)^2]}{\alpha^2} = \frac{\text{Var}(Y)}{\alpha^2}.$$ 

**Ex:** $Y$ is daily business cost, $E[Y] = 1500$, and $\text{Var}(Y) = \sigma^2 = (200)^2$.

$$P(Y \geq 2500) = P(Y - 1500 \geq 1000) = P(Y - \mu \geq 1000) \leq P(|Y - \mu| \geq 1000) \leq \frac{200^2}{1000^2} = \frac{5}{25}.$$ 

Cantelli's Inequality (one-sided Chebyshev)

If $\alpha > 0$, 
$$P(Y - \mu \geq \alpha) \leq \frac{\text{Var}(Y)}{\alpha^2}.$$ 

**Ex:** $\frac{1}{25}$