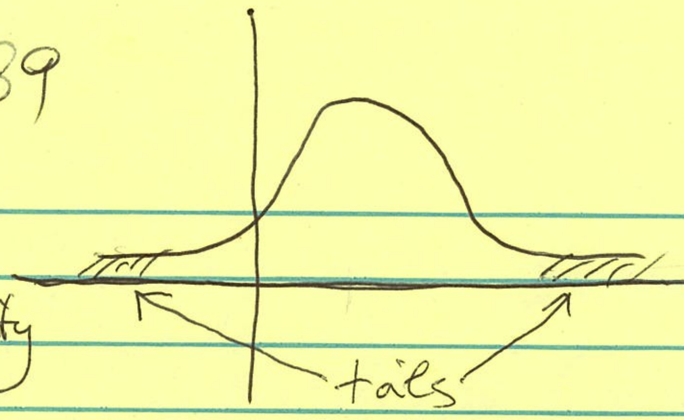
Tail Bounds:
Bound the probability of being far from the mean.



Ex: Expected business cost per day is \$1500. What is the probability that your cost on a particular day is > \$6000.

Markov's Inequality

Theorem: If X is a nonnegative r.v., then for any $\alpha > 0$, $P(X \geq \alpha) \leq E[X]/\alpha$.
Equivalently, $P(X \geq k E[X]) \leq 1/k$, for any $k > 0$.

Ex: If X is your cost on the particular day,
$P(X \geq 6000) \leq 1500/6000 = \frac{1}{4}$.
If $X = \begin{cases} 6000 & \text{with prob } 1/4 \\ 0 & \text{with prob } 3/4 \end{cases}$, this is best bound possible

Proof of theorem:

x is nonnegative

$$E[X] = \sum_x x p_X(x) = \sum_{x<\alpha} x p_X(x) + \sum_{x \geq \alpha} x p_X(x)$$

$$\geq 0 + \sum_{x \geq \alpha} \alpha p_X(x) = \alpha \sum_{x \geq \alpha} p_X(x) = \alpha P(X \geq \alpha),$$

so $P(X \geq \alpha) \leq E[X]/\alpha$

Chebyshev's Inequality: Suppose we also know Var(X).

Theorem: If Y is a r.v. with $E[Y] = \mu$, then for any $\alpha > 0$, $P(|Y-\mu| \geq \alpha) \leq Var(Y)/\alpha^2$.
Equivalently, if $\sigma = \sqrt{Var(Y)}$, then for any $t > 0$,
$$P(|Y-\mu| \geq t\sigma) \leq 1/t^2.$$

Proof: Let $X = (Y-\mu)^2$. X is nonnegative, so
$$P(|Y-\mu| \geq \alpha) = P(X \geq \alpha^2) \leq \frac{E[X]}{\alpha^2}$$
$$= \frac{E[(Y-\mu)^2]}{\alpha^2} = \frac{Var(Y)}{\alpha^2}$$

Ex: Y is daily business cost, $E[Y] = 1500$, and $Var(Y) = \sigma^2 = (200)^2$.
$$P(Y \geq 2500) = P(Y - 1500 \geq 1000) = P(Y-\mu \geq 1000)$$
$$\leq P(|Y-\mu| \geq 1000) \leq \frac{200^2}{1000^2} = \left(\frac{1}{5}\right)^2 = \frac{1}{25}$$

Cantelli's Inequality (one-sided Chebyshev)

If $\alpha > 0$, $P(Y-\mu \geq \alpha) \leq \dfrac{Var(Y)}{Var(Y) + \alpha^2}$.

Ex: $\frac{1}{26}$