Maximum Likelihood Estimators for the Normal Distribution $N(\theta_1, \theta_2)$. $(\theta_1 = \mu, \theta_2 = \sigma^2)$

Given independent samples $x_1, x_2, \ldots, x_n \sim N(\theta_1, \theta_2)$.

$$L(x_1, x_2, \ldots, x_n | \theta_1, \theta_2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} e^{-(x_i - \theta_1)^2/(2\theta_2)}$$

$$\ln L(x_1, \ldots, x_n | \theta_1, \theta_2) = \sum_{i=1}^{n} \left(-\frac{1}{2}\ln(2\pi\theta_2) - \frac{(x_i - \theta_1)^2}{2\theta_2} \right)$$

$$\frac{\partial}{\partial\theta_1} \ln L(x_1, \ldots, x_n | \theta_1, \theta_2) = \sum_{i=1}^{n} + \frac{2(x_i - \theta_1)}{2\theta_2} = \sum_{i=1}^{n} \frac{x_i - \theta_1}{\theta_2} = 0$$

$$\sum_{i=1}^{n} (x_i - \hat{\theta}_1) = 0$$

$$\sum_{i=1}^{n} x_i = n\hat{\theta}_1$$

$$\hat{\theta}_1 = \frac{1}{n}\sum_{i=1}^{n} x_i$$

To show that $\theta_1 = \hat{\theta}_1$ is a maximum:

$$\frac{\partial^2}{\partial\theta_1^2} \ln L(x_1, \ldots, x_n | \theta_1, \theta_2) = \sum_{i=1}^{n} -\frac{1}{\theta_2} < 0$$

so $\ln L$ is concave downward everywhere and $\theta_1 = \hat{\theta}_1$ is a maximum.

So the maximum likelihood estimator for $\theta_1 = \mu$ is the sample mean $\frac{1}{n}\sum_{i=1}^{n} x_i$.

$$\frac{\partial}{\partial\theta_2}\ln L(x_1,\dots,x_n|\theta_1,\theta_2) = \sum_{i=1}^{n}\left(-\frac{1}{2}\cdot\frac{2\pi}{2\pi\theta_2}+\frac{2(x_i-\theta_1)^2}{(2\theta_2)^2}\right)$$

$$=\sum_{i=1}^{n}\left(-\frac{1}{2\theta_2}+\frac{(x_i-\theta_1)^2}{2\theta_2^2}\right)=0$$

$$\sum_{i=1}^{n}\left(-\hat{\theta}_2+(x_i-\hat{\theta}_1)^2\right)=0$$

$$\sum_{i=1}^{n}(x_i-\hat{\theta}_1)^2=n\hat{\theta}_2$$

$$\hat{\theta}_2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\hat{\theta}_1)^2$$

Recall that $Var(Y)=E[(Y-\mu)^2]$, where $\mu=E[Y]$

$\hat{\theta}_2$ is called a <u>sample variance</u>.

To show that $\theta_2=\hat{\theta}_2$ ~~is~~ maximizes $\ln L$, ~~take~~ ~~compute~~ $\frac{\partial^2}{\partial\theta_2^2}\ln L$ and show that it is negative at $\theta_2=\hat{\theta}_2$.

In general, if there are two parameters θ_1 and θ_2 in the distribution, when you set the two derivatives to 0 you will get 2 equations in 2 unknowns, $\hat{\theta}_1$ and $\hat{\theta}_2$. Solve simultaneously.

Bias

~~Defn~~ Defn: An estimator $\hat{\theta}$ of θ is unbiased iff
$E[\hat{\theta}] = \theta$.

$\hat{\theta}$ is a function of $x_1, x_2, \cdots, x_n$. If we think of $x_1, \cdots, x_n$ as random variables, then $\hat{\theta}$ is also a random variable and it has an expected value. It's desirable that $E[\hat{\theta}] = \theta$.

For the estimator $\hat{\theta}_1$ of $N(\theta_1, \theta_2)$: $x_i \sim N(\theta_1, \theta_2)$

$$E[\hat{\theta}_1] = E\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[x_i] = \frac{1}{n}\sum_{i=1}^{n}\theta_1 = \frac{1}{n} n\theta_1$$

$= \theta_1$, so $\hat{\theta}_1$ is unbiased.