Proof: Let $X(s)$ and $Y(s)$ be the values of X and Y at outcome $s \in \Omega$.

$$E[X+Y] = \sum_{s\in\Omega} (X(s)+Y(s))P(s) = \sum_{s\in\Omega} (X(s)P(s) + Y(s)P(s))$$

$$= \sum_{s\in\Omega} X(s)P(s) + \sum_{s\in\Omega} Y(s)P(s) = E[X] + E[Y].$$

Ex: Let X be the number of heads when a coin with probability p of heads is flipped n times.

Let $X_i = \begin{cases} 1, & \text{if } i^{th} \text{ flip is heads} \\ 0, & \text{otherwise} \end{cases}$, for $1 \le i \le n$

"indicator random variables": value either 0 or 1.

$$E[X_i] = \underbrace{1 \cdot P(i^{th} \text{ flip is heads}) + 0 \cdot P(i^{th} \text{ flip is tails})}_{=p}$$

$$X = \sum_{i=1}^{n} X_i$$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] \quad \text{(linearity of exp)}$$

$$= \sum_{i=1}^{n} p = np$$

Ex: Shuffle 4 aces, pick 2 of them randomly and deal them face-down. Let X be the number of spades in these 2 piles. What is $E[X]$?

Let $X_i = \begin{cases} 1, & \text{if } i^{th} \text{ card is a spade} \\ 0, & \text{otherwise} \end{cases}$, for $i \in \{1,2\}$

$$X = X_1 + X_2$$

Are $X_1 = 1$ and $X_2 = 1$ independent?

$$P(X_2 = 1 \mid X_1 = 1) = 0 \neq P(X_2 = 1) = 1/4$$

$E[X_i] = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0) = \frac{1}{4}$, for $i \in \{1, 2\}$.

$E[X] = E[X_1 + X_2] = E[X_1] + E[X_2] = \frac{1}{2}$

Variance.

Consider two fair coin games between A and B.

1. A's gain per flip is $X = \begin{cases} +1, & \text{if heads} \\ -1, & \text{if tails} \end{cases}$
2. A's gain per flip is $Y = \begin{cases} +1000, & \text{if heads} \\ -1000, & \text{if tails} \end{cases}$

$E[X] = E[Y] = 0$, but are you equally happy playing either one? Too much variability in Y.

Defn: Let X be a r.v. with $E[X] = \mu$. The variance of X is $Var(X) = E[(X-\mu)^2]$, often denoted σ^2.

Defn: The standard deviation of X is $\sigma = \sqrt{Var(X)}$.

Ex: $Var(Y) = E[(Y-\mu)^2]$ where $\mu = E[Y] = 0$

$= 1000^2 \cdot \frac{1}{2} + (-1000)^2 \cdot \frac{1}{2} = 1{,}000{,}000$; $\sigma = 1000$

$Var(X) = 1$