Proof: Let $X(s)$ and $Y(s)$ be the values of $X$ and $Y$ at outcome $s \in \Omega$.

$$
E[X + Y] = \sum_{s \in \Omega} (X(s) + Y(s)) P(s) = \sum_{s \in \Omega} (X(s)P(s) + Y(s)P(s))
$$

$$
= \sum_{s \in \Omega} X(s)P(s) + \sum_{s \in \Omega} Y(s)P(s) = E[X] + E[Y].
$$

Ex: Let $X$ be the number of heads when a coin with probability $p$ of heads is flipped $n$ times.

Let $X_i = 1$, if $i^{th}$ flip is heads, for $1 \leq i \leq n$.

"indicator random variables" : value either 0 or 1.

$$
E[X] = 1 \cdot P(i^{th} \text{ flip is heads}) + 0 \cdot P(i^{th} \text{ flip is tails})
$$

$$
= p
$$

$$
X = \sum_{i=1}^{n} X_i
$$

$$
E[X] = E\left[ \sum_{i=1}^{n} X_i \right] = \sum_{i=1}^{n} E[X_i] \quad \text{(linearity of } E) \quad \sum_{i=1}^{n} p = np
$$

Ex: Shuffle 4 aces, pick 2 of them randomly and deal them face-down. Let $X$ be the number of spades in those 2 piles. What is $E[X]$?

Let $X_i = 1$, if $i^{th}$ card is a spade, for $i \in \{1, 2\}$.

$$
X = X_1 + X_2
$$

Are $X_1 = 1$ and $X_2 = 1$ independent?

$$
P(X_2 = 1 | X_1 = 1) = 0 \neq P(X_2 = 1) = \frac{1}{4}
$$
\[ E[X_i] = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0) = \frac{1}{4}, \text{ for } i \in \{1, 2\} \]
\[ E(X) = E[X_1 + X_2] = E[X_1] + E[X_2] = \frac{1}{2} \]

Variance:
Consider two fair coin games between A and B.
1. A's gain per flip is \( X = 3 + 1 \), if heads
   \[ E[X] = 3 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 2 \]
2. A's gain per flip is \( Y = 3 + 1000 \), if heads
   \[ E[Y] = 3 \cdot \frac{1}{2} + 1000 \cdot \frac{1}{2} = 500 \]

\[ E[X] = E[Y] = 0, \text{ but are you equally happy playing either one? Too much variability in } Y. \]

Defn: Let \( X \) be a r.v. with \( E[X] = \mu \). The variance of \( X \) is \( \text{Var}(X) = E[(X - \mu)^2] \), often denoted \( \sigma^2 \).
Defn: The standard deviation of \( X \) is \( \sigma = \sqrt{\text{Var}(X)} \).

\[ \text{Var}(Y) = E[(Y - \mu)^2] \]
where \( \mu = E[Y] = 0 \)
\[ = 1000^2 \cdot \frac{1}{2} + (-1000)^2 \cdot \frac{1}{2} = 1,000,000 \]
\[ \sigma = 1000 \]

\[ \text{Var}(X) = 1 \]