Examples on slidepack 6, slide 13.

$$E[X] = \sum_{a=0}^{10} a\, p_X(a) = \sum_{a=0}^{10} a\, P(X=a)$$

$$= \sum_{a=0}^{10} a \cancel{\frac{1}{2^{10}}\binom{10}{a}} \binom{10}{a} p^a (1-p)^{10-a}$$

$$= 1\cdot\binom{10}{1} p^1 (1-p)^9 + 2\binom{10}{2} p^2 (1-p)^8 + \cdots,$$

Does this have a nice closed form? Yes, coming soon.

Ex: Let X be the number of flips up to and including the first head, when a coin with probability p of heads is flipped independently.

(Geometric random variable)

$$p_X(1) = P(X=1) = p$$

$$p_X(2) = P(X=2) = (1-p)p$$

$$p_X(i) = P(X=i) = (1-p)^{i-1} p$$

$$E[X] = \sum_{i=1}^{\infty} i\, p_X(i) = \sum_{i=1}^{\infty} i(1-p)^{i-1} p = p \sum_{i=1}^{\infty} i(1-p)^{i-1}$$

Calculus: $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x} = (1-x)^{-1}$, if $|x|<1$

Differentiate: $\sum_{i=0}^{\infty} i x^{i-1} = \cancel{+}(1-x)^{-2} = \frac{1}{(1-x)^2}$

$x = 1-p$: $E[X] = p \cdot \frac{1}{(1-(1-p))^2} = \boxed{\frac{1}{p}}$

$p = \frac{1}{2} \Rightarrow E[X] = 2$

$p = \frac{1}{10} \Rightarrow E[X] = 10$

$$\sum_{i=0}^{\infty} i x^{i-1} = \sum_{i=1}^{\infty} i x^{i-1}$$

Linearity of Expectation

Defn: If X is a r.v., $E[g(X)] = \sum_a g(a) p_X(a)$

Theorem 1: For any constants a and b,
$E[aX+b] = aE[X]+b$.

Proof: $E[aX+b] = \sum_i (ai+b)p_X(i)$ ~~$+ \sum_i (ai p_X(i))$~~

$$= \sum_i (ai\, p_X(i) + b\,p_X(i))$$

$$= \sum_i ai\, p_X(i) + \sum_i b\,p_X(i)$$

$$= a\sum_i i\, p_X(i) + b\sum_i p_X(i)$$

$$= a\,E[X]+b$$

Ex: A casino charges \$1 to play the following game. They flip a coin with probability 1/8 of heads until it comes up heads and they pay you 12¢ for each flip. Let X be the number of flips. Do you expect to win or lose money?

$E[12X-100] = 12E[X]-100 = 12\cdot 8-100 = -4$.

Theorem 2: Let X and Y be two random variables, possibly dependent. Then $E[X+Y] = E[X]+E[Y]$.

Linearity is special. In general,

$$E[XY] \neq E[X]E[Y],$$
$$E[X^2] \neq (E[X])^2,$$
$$E[\sqrt{X}] \neq \sqrt{E[X]}, \text{ etc.}$$