Examples on slide pack 6, slide 13.

\[ E[X] = \sum_{a=0}^{10} a P(X=a) = \sum_{a=0}^{10} a P(X=a) \]

\[ = \sum_{a=0}^{10} a (\binom{10}{a} p^a (1-p)^{10-a}) = 1 \cdot (10) p^0 (1-p)^9 + 2 \binom{10}{2} p^2 (1-p)^8 + \ldots \]

Does this have a nice closed form? Yes, coming soon.

**Example:** Let \( X \) be the number of flips up to and including the first head when a coin with probability \( p \) of heads is flipped independently.

Geometric random variable:

\[ P_X(1) = P(X=1) = p \]
\[ P_X(2) = P(X=2) = (1-p)p \]
\[ P_X(3) = P(X=3) = (1-p)^2 p \]
\[ \ldots \]

\[ E[X] = \sum_{i=1}^{\infty} i P_X(i) = \sum_{i=1}^{\infty} i (1-p)^{i-1} p = p \sum_{i=1}^{\infty} i (1-p)^{i-1} \]

Calculus:

\[ \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}, \text{ if } \left| x \right| < 1 \]

Differentiate:

\[ \sum_{i=0}^{\infty} i x^{i-1} = \frac{1}{(1-x)^2} \]

\( x = 1-p \):

\[ E[X] = p \cdot \frac{1}{(1-(1-p))^2} = \frac{1}{p} \]

\( p = \frac{1}{2} \Rightarrow E[X] = 2 \]
\( p = \frac{1}{10} \Rightarrow E[X] = 10 \]
Linearity of Expectation

Defn: If \( X \) is a r.v., \( E[g(X)] = \sum_a g(a)P_X(a) \)

**Theorem 1:** For any constants \( a \) and \( b \),
\[
E[aX+b] = aE[X]+b.
\]

**Proof:**
\[
E[aX+b] = \sum_a (aX+b)P_X(a) = \sum_a aX(a)P_X(a) + \sum_a bP_X(a) = a\sum_a XP_X(a) + b\sum_a P_X(a) = aE[X]+b
\]

\( \sum_a P_X(a) = 1 \)

Ex: A casino charges $1 to play the following game. They flip a coin with probability \( \frac{1}{8} \) of heads until it comes up heads and they pay you \( 12X \) for each flip. Let \( X \) be the number of flips. Do you expect to win or lose money?
\[
E[12X-100] = 12E[X]-100 = 12 \cdot \frac{1}{8} - 100 = -4.
\]

**Theorem 2:** Let \( X \) and \( Y \) be two random variables, possibly dependent. Then \( E[X+Y] = E[X]+E[Y] \).

Linearity is special. In general,
\[
E[XY] \neq E[X]E[Y], \quad E[X^2] \neq (E[X])^2, \quad E[\sqrt{X}] \neq \sqrt{E[X]}, \quad \text{etc.}
\]