For instance, prior $p = \frac{1}{2} \Rightarrow$ posterior $\frac{2p}{1+p} = \frac{2}{3}$

<u>Defn</u>: Two events E and F are <u>independent</u> iff
$P(E \cap F) = P(E)P(F)$. Otherwise, they are dependent.

<u>Ex</u>: Roll 2 fair dice, yielding outcomes D_1 and D_2.
Let E be "$D_1 = 1$", F be "$D_1 + D_2 = 7$", G be "$D_1 + D_2 = 5$".
$P(E) = \frac{1}{6}$, $P(F) = \frac{6}{36} = \frac{1}{6}$,
$P(E \cap F) = P(D_1 = 1 \cap D_2 = 6) = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6} = P(E)P(F)$,
so E and F are independent.
$P(G) = \frac{4}{36} = \frac{1}{9}$
$P(E \cap G) = P(D_1 = 1 \cap D_2 = 4) = \frac{1}{36} \neq \frac{1}{6} \cdot \frac{1}{9} = P(E)P(G)$,
so E and G are dependent.

<u>Defn</u>: $E_1, E_2, \cdots, E_n$ are <u>independent</u> iff for every subset S of $\{1, 2, \cdots, n\}$,
$$P\left(\bigcap_{i \in S} E_i\right) = \prod_{i \in S} P(E_i).$$

<u>Ex</u>: Let X and Y each be ± 1 with equal probability and independently of each other.
Let E be "$X = +1$", F be "$Y = +1$", G be "$XY = +1$".
These are pairwise independent, but
$P(E \cap F \cap G) = P(X = +1 \cap Y = +1) = \frac{1}{4} \neq \frac{1}{8} = P(E)P(F)P(G)$
so E, F, G are not independent.

<u>Theorem</u>: If $P(F) > 0$, then E and F are independent iff $P(E|F) = P(E)$.

<u>Proof</u>: $\Rightarrow$: Suppose E and F are independent. Then
$P(E)P(F) = P(E \cap F) = P(E|F)P(F)$, so divide by $P(F)$.
$\Leftarrow$: Suppose $P(E|F) = P(E)$. Then

$P(E \cap F) = P(E|F)P(F) = P(E)P(F)$, So
E and F are independent.

Ex: Suppose a biased coin comes up heads with probability p. Suppose it is flipped n times independently.

$P(n \text{ heads}) = p^n$

$P(\text{first } k \text{ are heads and last } n-k \text{ are tails}) = p^k(1-p)^{n-k}$

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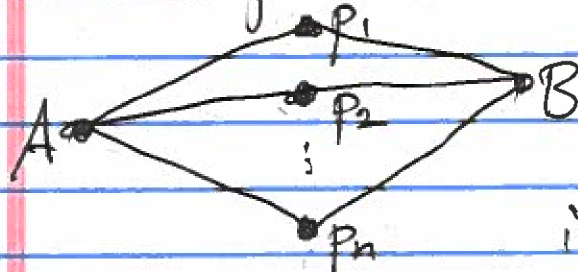
$P(k \text{ heads and } n-k \text{ tails}) = \binom{n}{k} p^k (1-p)^{n-k}$.

$P(0 \text{ heads or } 1 \text{ head or} \cdots \text{or } n \text{ heads})$

$$= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + (1-p))^n = 1^n = 1,$$

as it must be, since it's the whole sample space.

Ex: Network failure.

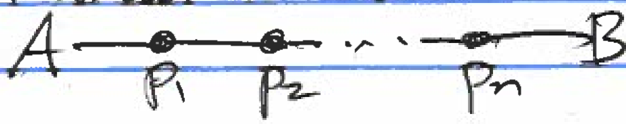


Suppose n components in parallel can fail with probabilities $p_1, p_2, \ldots, p_n$ independently.

$P(A \text{ can communicate with } B) = 1 - P(\text{all } n \text{ fail})$

$= 1 - p_1 p_2 \cdots p_n$.

What about in series?



$P(A \text{ can communicate with } B) = P(\text{none of } n \text{ fail})$

$= (1-p_1)(1-p_2)\cdots(1-p_n)$.