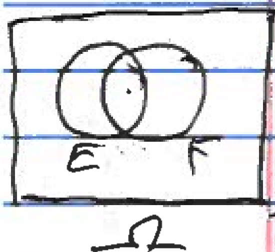
Generalized Chain Rule: $P(E_1 \cap E_2 \cap \cdots \cap E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1 \cap E_2)\cdots P(E_n|E_1 \cap E_2 \cap \cdots \cap E_{n-1})$

Proof: Choose $E = E_n$ and $F = E_1 \cap E_2 \cap \cdots \cap E_{n-1}$ and use induction on n.

Law of Total Probability: If E and F are events, then $P(E) = P(E|F)P(F) + P(E|\bar{F})P(\bar{F})$.

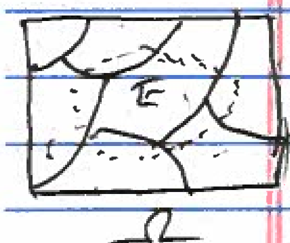


Proof:
$$\begin{aligned} P(E) &= P((E \cap F) \cup (E \cap \bar{F})) \\ &= P(E \cap F) + P(E \cap \bar{F}) \quad \text{(Axiom 3)} \\ &= P(E|F)P(F) + P(E|\bar{F})P(\bar{F}). \end{aligned}$$

Ex: Sally will take either Physics or Chemistry. She will get an A in Physics with probability 3/4 and an A in Chemistry with probability 3/5. She flips a fair coin to decide which course to take.

$$P(A) = P(A|\text{Phys})P(\text{Phys}) + P(A|\text{Chem})P(\text{Chem})$$
$$= \frac{3}{4}\cdot\frac{1}{2} + \frac{3}{5}\cdot\frac{1}{2} = \frac{27}{40}.$$

General LTP: If $F_1, F_2, \ldots, F_n$ partition Ω, then $P(E) = \sum_{i=1}^{n} P(E|F_i)P(F_i)$.



Bayes' Theorem (Rev. Thomas Bayes, c. 1701–1761):
$$P(F|E) = \frac{P(E|F)P(F)}{P(E)}.$$

Allows us to reverse the conditioning.

Proof:
$$P(F|E) = \frac{P(E \cap F)}{P(E)} \quad \text{(defn of cond. prob.)}$$
$$= \frac{P(E|F)P(F)}{P(E)} \quad \text{(chain rule)}$$

Corollary:
$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|\bar{F})P(\bar{F})}.$$

Ex: 60% of email is spam.
90% of spam has a forged header.
20% of nonspam has a forged header.
Let F = forged header and J = spam.
What is $P(J|F)$?

$$P(J|F)=\frac{P(F|J)P(J)}{P(F|J)P(J)+P(F|\bar{J})P(\bar{J})} \quad \text{(Bayes)}$$

$$=\frac{0.9\times 0.6}{0.9\times 0.6+0.2\times(1-0.6)}\approx 0.871$$

prior $=\underset{P(J)}{0.6}$, posterior $\approx \underset{P(J|F)}{0.871}$

Ex: Paternity testing
Child has (A,a) gene pair (event B_{Aa})
Mother has (A,A)
Two possible fathers: $F_1=(a,a)$, $F_2=(A,a)$.
$P(F_1)=p$, $P(F_2)=1-p$

$$P(F_1|B_{Aa})=\frac{P(B_{Aa}|F_1)P(F_1)}{P(B_{Aa}|F_1)P(F_1)+(B_{Aa}|F_2)P(F_2)}$$

$$=\frac{1\cdot p}{1\cdot p+\frac{1}{2}(1-p)}=\frac{2p}{1+p}\geq p$$