$\binom{n}{k} = C(n,k) = \dfrac{n!}{k!(n-k)!}$

"number of combinations of n objects taken k at a time"

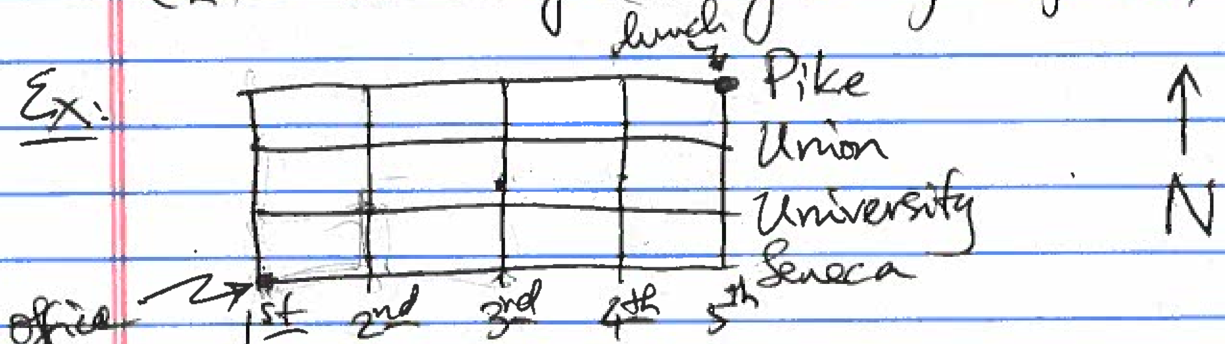
"n choose k"

"binomial coefficients"

$P(n,k)$: number of sequences of k objects from $1,2,\cdots,n$.

$\binom{n}{k}$: number of sets of k objects from $1,2,\cdots,n$

Ex:



How many routes from 1st & Seneca to 5th & Pike that go either N or E at each intersection?

N N E E N E ~~E~~

$\dfrac{7!}{3!\,4!} = \binom{7}{3} = \binom{7}{4}$

Identities:

(1) $\binom{n}{k} = \binom{n}{n-k}$, where $0 \le k \le n$.

(2) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, where $1 \le k \le n-1$

Consider the first of the n objects, say x. Either

(a) x is in the subset of size k ($\binom{n-1}{k-1}$ ways to finish), or

(b) x is not in the subset of size k ($\binom{n-1}{k}$ ways to finish).

(3) <u>Binomial Theorem</u>: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Proof by counting.

$$(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ factors}}$$

<u>Ex</u>:
$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)(x+y) \\ &= xxx + xxy + xyx + xyy \\ &\quad + yxx + yxy + yyx + yyy \\ &= x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

In general, choose either x or y from each of the n factors and multiply these n choices. How many of these terms look like $x^k y^{n-k}$? There are $\binom{n}{k}$ ways to choose x in k of the factors.

$$\underbrace{\underline{x}\ \underline{y}\ \underline{y}\ \underline{x}\ \underline{x}\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }}_{n}$$

<u>Corollary</u>: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

<u>Proof</u>: $2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$

<u>Complementing</u>

<u>Ex</u>: Assume ♠J is face-up trump on the table. How many 5-card Schnapsen hands contain ≥ 1 trump?

$\binom{4}{1}\binom{18}{4}$ No! Overcounting.

♠Q ♠K

♠K ♠Q