CSE 312: Foundations of Computing II
Quiz Section #9: Moment Generating Function and Covariance
(solutions)

Review: Main Theorems and Concepts

Moment Generating Function: Let X be a real valued random variable. If M is the moment generating function of X, then $M(t) = \mathrm{E}\left[e^{tX}\right]$.

Moment Generating Functions to Distributions: If random variables X, Y have the same moment generating function, then they have the same cumulative distribution function.

Covariance: For random variables X and Y, the covariance is defined as $\mathsf{Cov}(X, Y) =$

$$\mathrm{E}[(X - \mathrm{E}[X])(Y - \mathrm{E}[Y])] = \mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y].$$

Exercises

1. Let X_1 and X_2 be independent standard normal (mean 0 and variance 1) random variables. In class, we used the moment generating function to prove that $X_1 + X_2$ is distributed according to a normal with mean 0 and variance 2. Now, show the same fact by explicitly computing the PDF of $X_1 + X_2$.

   Let f_1 and f_2 be the PDFs of X_1 and X_2 respectively. Denote $X = X_1 + X_2$, and let f be the PDF of X. Our starting point is the following equality:

$$f(x) = \int_{-\infty}^{\infty} f_1(x - x') f_2(x') dx'.$$

   This is true because, for every value of x', when $X_2 = x'$, $X_1 + X_2 = x$ for $X_2 = x = x'$. In other words, X takes the value x by setting $X_2 = x'$ and $X_1 = x - x'$. We have the product of the PDFs since X_1 and X_2 are independent. We proceed to calculate $f(x)$.

$$
\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} f_1(x-x')f_2(x')dx' \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x-x')^2/2} \frac{1}{\sqrt{2\pi}} e^{-x'^2/2} dx' \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(x^2+x'^2-2xx'+x'^2)/2} dx' \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(x^2+2x'^2-2xx')/2} dx' \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(x^2/2+2(x'-x/2)^2)/2} dx' \\
&= \frac{e^{-x^2/4}}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x'-x/2)^2} dx'.
\end{aligned}
$$

Now observe that $\frac{e^{-(x'-x/2)^2}}{\sqrt{\pi}}$ is the PDF of a Gaussian with mean $x/2$ and variance $\frac{1}{2}$. This implies that $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x'-x/2)^2} dx' = \frac{1}{\sqrt{2}}$. We can now conclude that

$$
f(x) = \frac{e^{-x^2/(2\cdot 2)}}{\sqrt{2\pi} \cdot \sqrt{2}},
$$

which is the PDF of a normal distribution with mean 0 and variance 2.

2. Compute the moment generating function of the uniform distribution on $[a, b]$.

The pdf of the uniform distribution is given by

$$
f(x) = \begin{cases} 1/(b-a), & a \le x \le b \\ 0, & \text{otherwise.} \end{cases}
$$

By definition, we know that $M(t) = \mathrm{E}[e^{tX}]$, where X is distributed according to the uniform distribution. We now proceed to do the calculations:

$$
\begin{aligned}
M(t) &= \int_a^b e^{tx}/(b-a) dx \\
&= \frac{1}{b-a} \cdot \int_a^b e^{tx} dx \\
&= \frac{1}{t(b-a)} \cdot (e^{tx})\Big|_{x=a}^{b} \\
&= \frac{e^{bt} - e^{at}}{t(b-a)}.
\end{aligned}
$$

3. X is known to be a discrete distribution. In addition, we know that the moment generation function of X is given by

$$M(t) = e^{-2t}/4 + e^{-t}/6 + 1/4 + e^{t}/6 + e^{2t}/6.$$

Compute the probability that $|X| \le 1$.

Using the fact that the same moment generating function implies the same cumulative distribution function, and the fact that it is a discrete distribution, we can conclude that X is supported on $\{-2, -1, 0, 1, 2\}$ with $p(X = -2) = 1/4$, $p(X = -1) = 1/6$, $p(X = 0) = 1/4$, $p(X = 1) = 1/6$ and $p(X = 2) = 1/6$. Therefore, $p(|X| \le 1) = 1/6 + 1/4 + 1/6 = 7/12$.

4. (a) Let X and Y be random variables. Show that $\mathsf{Var}(X+Y) = \mathsf{Var}(X)+\mathsf{Var}(Y)+2\mathsf{Cov}(X, Y)$.

We have

$$\begin{aligned}
\mathsf{Var}(X + Y) &= \mathrm{E}[(X + Y)^2] - \mathrm{E}[X + Y]^2 \\
&= \mathrm{E}[X^2 + Y^2 + 2XY] - \mathrm{E}[X]^2 - \mathrm{E}[Y]^2 - 2\mathrm{E}[X]\mathrm{E}[Y] \\
&= \mathrm{E}[X^2] - \mathrm{E}[X]^2 + \mathrm{E}[Y^2] - \mathrm{E}[Y]^2 + 2(\mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y]) \\
&= \mathsf{Var}[X] + \mathsf{Var}[Y] + 2\mathsf{Cov}(X, Y).
\end{aligned}$$

(b) A fair die is rolled n times. Let X be the number of 1's and let Y be the number of 6's. Compute $\mathsf{Cov}(X, Y)$.

We use the expression from Part (a) to solve this question. First, note that X and Y are distributed according to a Binomial distribution with parameter 1/6. In addition, $X + Y$ is distributed according to a Binomial distribution with parameter 1/3. Therefore

$$\mathsf{Var}[X] = \mathsf{Var}[Y] = n \cdot 1/6 \cdot 5/6 = 5n/36,$$

and

$$\mathsf{Var}[X + Y] = n \cdot 1/3 \cdot 2/3 = 2n/9.$$

From Part (a), we know that

$$2\mathsf{Cov}(X, Y) = \mathsf{Var}[X + Y] - \mathsf{Var}[X] - \mathsf{Var}[Y] = -n/18.$$

Hence $\mathsf{Cov}(X, Y) = -n/36$.

5. (a) Let X, Y, Z be random variables. Show that $\mathsf{Cov}(X + Y, Z) = \mathsf{Cov}(X, Z) + \mathsf{Cov}(Y, Z)$.

We have,

$$\begin{aligned}\mathsf{Cov}(X+Y,Z) &= \mathrm{E}[(X+Y)Z] - \mathrm{E}[X+Y]\mathrm{E}[Z] \\ &= \mathrm{E}[XZ] + \mathrm{E}[YZ] - \mathrm{E}[X]\mathrm{E}[Z] - \mathrm{E}[X]\mathrm{E}[Z] \\ &= \mathsf{Cov}(X,Z) + \mathsf{Cov}(Y,Z).\end{aligned}$$

(b) Define random variables $X = X_1 + \ldots + X_n$ and $Y = Y_1 + \ldots + Y_n$. Show that $\mathsf{Cov}(X,Y) = \sum_{i,j=1}^{n} \mathsf{Cov}(X_i, Y_j)$.

From Part (a),

$$\begin{aligned}\mathsf{Cov}(X,,Y) &= \mathsf{Cov}(X_1,Y) + \mathsf{Cov}(X_2 + \ldots + X_n, Y) \\ &= \mathsf{Cov}(X_1,Y) + \mathsf{Cov}(X_2,Y) + \ldots + \mathsf{Cov}(X_n,Y) \\ &= \sum_{i,j=1}^{n} \mathsf{Cov}(X_i,Y_j).\end{aligned}$$

6. Let X_1 and X_2 be independent standard normal (mean 0 and variance 1) random variables. If $Y_1 = 2X_1 + X_2$ and $Y_2 = X_1 - X_2$, then compute $\mathsf{Cov}(Y_1, Y_2)$.

By definition,

$$\begin{aligned}\mathsf{Cov}(Y_1,Y_2) &= \mathrm{E}[(2X_1+X_2)(X_1-X_2)] - \mathrm{E}[2X_1+X_2]\mathrm{E}[X_1-X_2] \\ &= \mathrm{E}[(2X_1+X_2)(X_1-X_2)] \\ &= \mathrm{E}[2X_1^2] + \mathrm{E}[X_1X_2] - \mathrm{E}[X_2^2],\end{aligned}$$

where the second equality follows from the fact that $\mathrm{E}[X_1 - X_2] = 0$. Since X_1 and X_2 are independent, and $\mathrm{E}[X_1] = \mathrm{E}[X_2] = 0$,

$$\mathrm{E}[2X_1^2] + \mathrm{E}[X_1X_2] - \mathrm{E}[X_2^2] = 2\mathrm{E}[X_1^2] - \mathrm{E}[X_2^2].$$

We now proceed to compute $\mathrm{E}[X_1^2], \mathrm{E}[X_2^2]$.

$$\mathrm{E}[X_1^2] = \mathrm{E}[X_2^2] = \mathsf{Var}[X_1] - \mathrm{E}[X_1]^2 = \mathsf{Var}[X_1] = 1.$$

Therefore, $\mathsf{Cov}(Y_1,Y_2) = 1$.