Lecture 12: Expectation, and Linearity of Expectation

Anup Rao

April 26, 2019

We discuss the expectation of a real valued random variable.

Expectation

When a random variable takes a number as a value, it makes sense to talk about the average value that it takes. The *expected value* of a random variable is defined:

$$\mathbb{E}\left[X\right] = \sum_x p(X = x) \cdot x. \tag{1}$$

For example, let X be the roll of a 6-sided die. The expectation

$$\mathbb{E}\left[X\right] = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = \frac{21}{6} = 3.5.$$

If X is the number of heads when we toss a coin n times, the expected value of X is

$$\mathbb{E}\left[X\right] = \frac{\binom{n}{0}}{2^n} \cdot 0 + \frac{\binom{n}{1}}{2^n} \cdot 1 + \ldots + \frac{\binom{n}{n}}{2^n} \cdot n. \tag{2}$$

$\mathbb{E}\left[X\right]$ gives the *center* of mass of the distribution of X. However, there are many different distributions that can have the same expectation.

For example, let X, Y, Z be random variables such that

$$X = \begin{cases} 1000 & \text{with probability } 1/2, \\ -1000 & \text{with probability } 1/2. \end{cases}$$

$$Y = \begin{cases} 1 & \text{with probability } \frac{n-1}{n}, \\ -(n-1) & \text{with probability } \frac{1}{n}. \end{cases}$$

$$Z = \begin{cases} 0 & \text{with probability } 1. \end{cases}$$

Then note that $\mathbb{E}\left[X\right] = \mathbb{E}\left[Y\right] = \mathbb{E}\left[Z\right] = 0$, even though these three variables have vastly different distributions. There are several misconceptions that people have about expectations.

- It is not necessarily true that a random variable will be close to its expectation with high probability. For example, X as defined above is never close to 0.
- It is not necessarily true that a random variable will be above its expectation with probability about half and below with probability half. Consider Y above.

Linearity of Expectation

THE FORMULA FOR EXPECTATION GIVEN IN (1) is not always the easiest way to calculate the expectation. Here are some observations that can make it much easier to calculate the expectation.

The first extremely useful concept is the notion of linearity of expectation.

Fact 1. *If X and Y are real valued random variables in the same probability space, then* $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.

Proof.

$$\begin{aligned}
\mathbb{E}[X+Y] &= \sum_z p(X+Y=z)\cdot z \\
&= \sum_{x,y} p(X=x, Y=y)\cdot(x+y) \\
&= \sum_{x,y} p(X=x, Y=y)\cdot x + \sum_{x,y} p(X=x, Y=y)\cdot y.
\end{aligned}$$

Now we can express the first term

$$\begin{aligned}
\sum_{x,y} p(X=x, Y=y)\cdot x &= \sum_{x,y} p(X=x)\cdot x\cdot p(Y=y|X=x) \\
&= \sum_x p(X=x)\cdot x\left(\sum_y p(Y=y|X=x)\right) \\
&= \sum_x p(X=x)\cdot x = \mathbb{E}[X].
\end{aligned}$$

Similarly, the second term is $\mathbb{E}[Y]$. □

More generally, we have that for any real numbers α, β,

$$\mathbb{E}[\alpha X + \beta Y] = \alpha\cdot\mathbb{E}[X] + \beta\cdot\mathbb{E}[Y].$$

The amazing thing is that linearity of expectation even works when the random variables are dependent. This does not hold, for example, with multiplication—in general $\mathbb{E}[X\cdot Y] \neq \mathbb{E}[X]\cdot\mathbb{E}[Y]$.

However, when X, Y are independent, we do have $\mathbb{E}[X\cdot Y] = \mathbb{E}[X]\cdot\mathbb{E}[Y]$.

Example: Expected Number of Heads

Suppose you toss a coin n times. Let X denote the total number of heads. Define X_i by

$$X_i = \begin{cases} 1 & \text{if the } i\text{'th toss gives heads,} \\ 0 & \text{otherwise.} \end{cases}$$

Then we see that $X = X_1 + \ldots + X_n$. Moreover,

$$\mathbb{E}[X_i] = (1/2)0 + (1/2)1 = 1/2.$$

So, we get

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \ldots + \mathbb{E}[X_n] = n/2.$$

Example: Birthday Paradox

There are 96 students enrolled in CSE312. Assuming that each of them has a uniformly random and independent birthday, how many pairs of students are expected to have the same birthday?

For $i < j$, let $X_{i,j}$ be the random variable such that

$$X_{i,j} = \begin{cases} 1 & \text{if } i, j \text{ share a birthday} \\ 0 & \text{otherwise.} \end{cases}$$

Then the number of pairs of students with the same birthday is just $\mathbb{E}\left[\sum_{i<j\in[96]} X_{i,j}\right] = \sum_{i<j\in[96]} \mathbb{E}\left[X_{i,j}\right]$, by linearity of expectation.

We have $\mathbb{E}\left[X_{i,j}\right] = \frac{1}{365}$, so the expected number of pairs with the same birthday is:

$$\sum_{i<j\in[96]} \mathbb{E}\left[X_{i,j}\right] = \frac{\binom{96}{2}}{365} = 12.49\ldots$$