CSE 312

Foundations of Computing II

Lecture 8: Bayes Rule, Limited Independence

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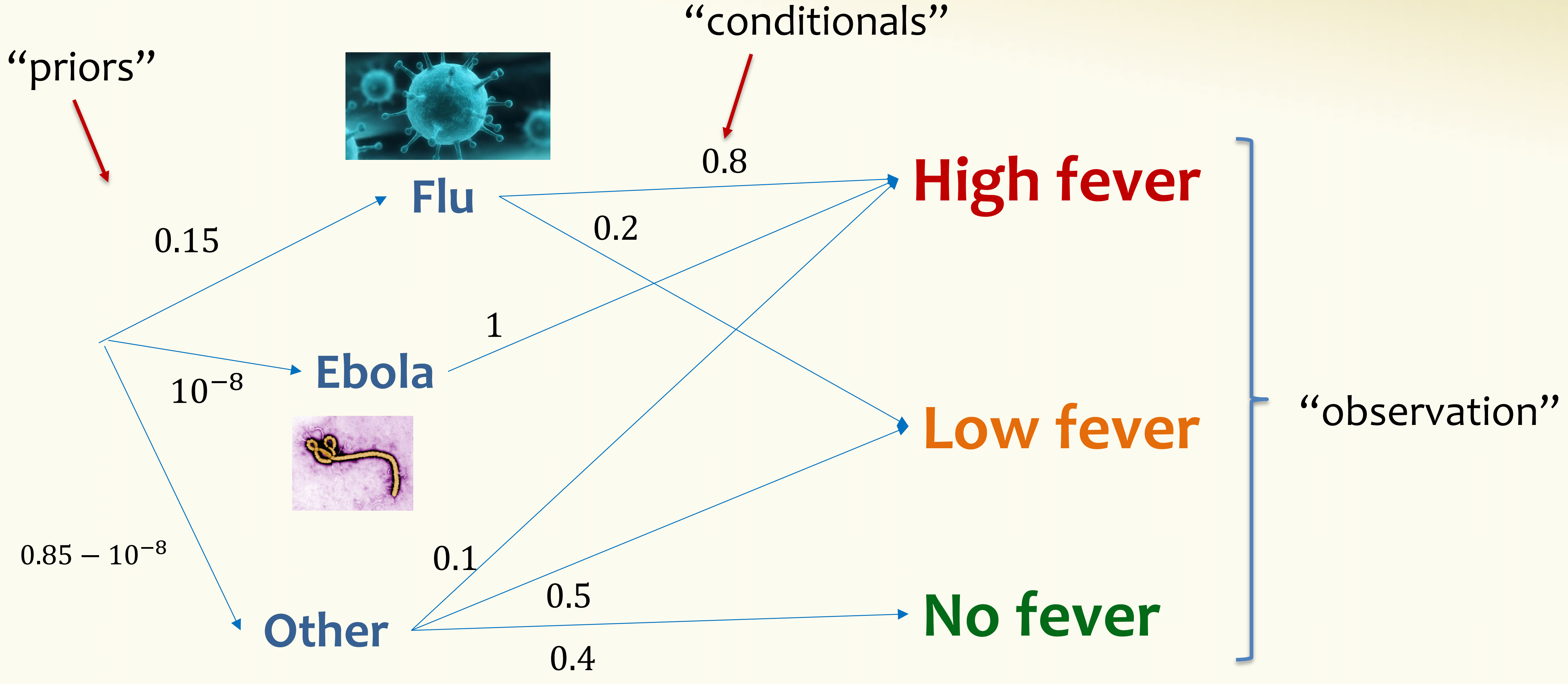
Today

- Bayes Rule
- Independence of multiple events

On LaTeX

- Overleaf is not the best approach for using LaTeX
  – Tool for collaborative editing of LaTeX documents.
  – Not needed for class.
  – Has become somewhat unstable.
- LaTeX is free software – you can find several installations, depending on OS.
- Several environment for LaTeX development, your favorite editor often will do.

7.1 – Bayes Rule



Assume we observe high fever, what is the probability that the subject has Ebola?

Posterior: $\mathbb{P}(\textbf{Ebola}|\textbf{High fever})$

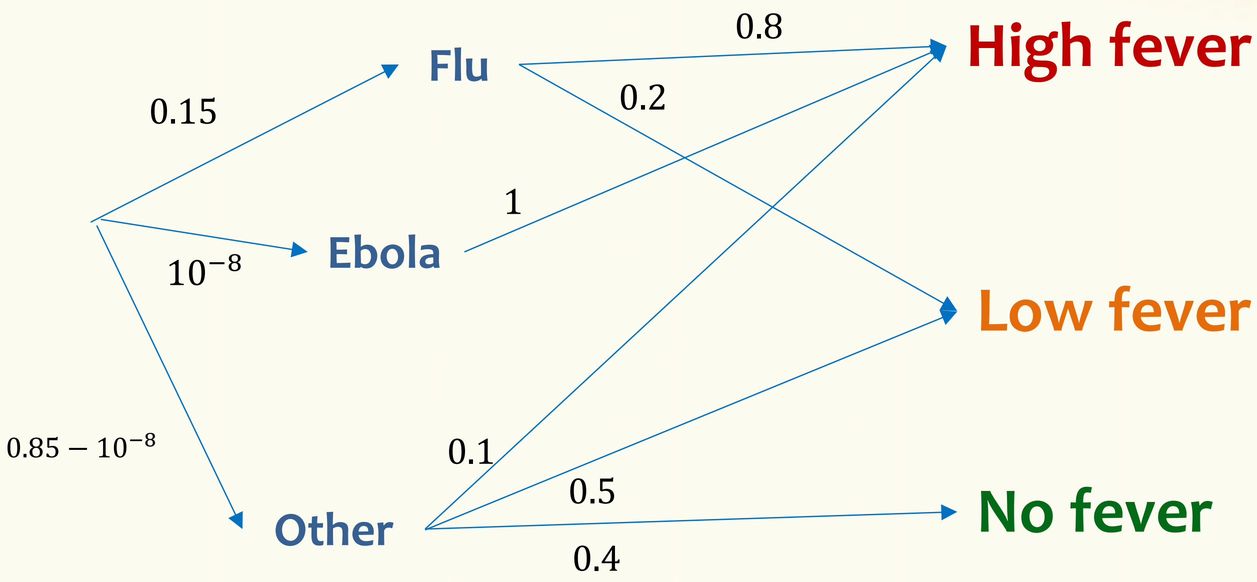
Bayes Rule

Theorem. (Bayes Rule) For events $\mathcal{A}$ and $\mathcal{B}$, where $\mathbb{P}(\mathcal{A}), \mathbb{P}(\mathcal{B}) > 0$,

$$\mathbb{P}(\mathcal{B}|\mathcal{A}) = \frac{\mathbb{P}(\mathcal{B}) \cdot \mathbb{P}(\mathcal{A}|\mathcal{B})}{\mathbb{P}(\mathcal{A})}$$

Rev. Thomas Bayes [1701-1761]

Proof: $\mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\mathcal{B}|\mathcal{A}) = \mathbb{P}(\mathcal{A} \cap \mathcal{B})$



$$\mathbb{P}(\text{Ebola}|\text{High fever}) = \frac{\mathbb{P}(\text{Ebola}) \cdot \mathbb{P}(\text{High fever}|\text{Ebola})}{\mathbb{P}(\text{High fever})}$$

$$= \frac{10^{-8} \cdot 1}{0.15 \times 0.8 + 10^{-8} \times 1 + (0.85 - 10^{-8}) \times 0.1} \approx 7.4 \times 10^{-8}$$

$$\mathbb{P}(\text{Flu}|\text{High fever}) \approx 0.89$$

Most-likely a-posteriori outcome (MLA)

$$\mathbb{P}(\text{Other}|\text{High fever}) \approx 0.11$$

Bayes Rule – Example

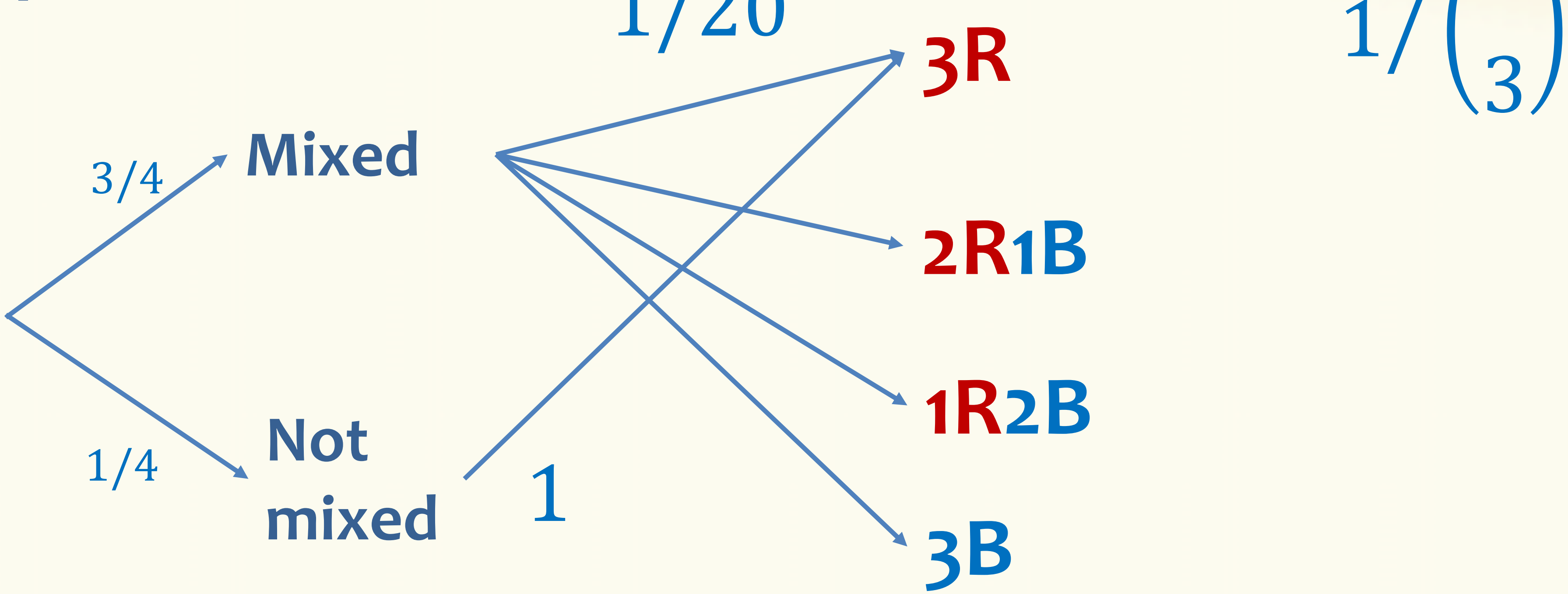
Setting: An urn contains 6 balls:

- 3 **red** and 3 **blue** balls w/ probability ¾
- 6 **red** balls w/ probability ¼

We draw three balls at random from the urn.

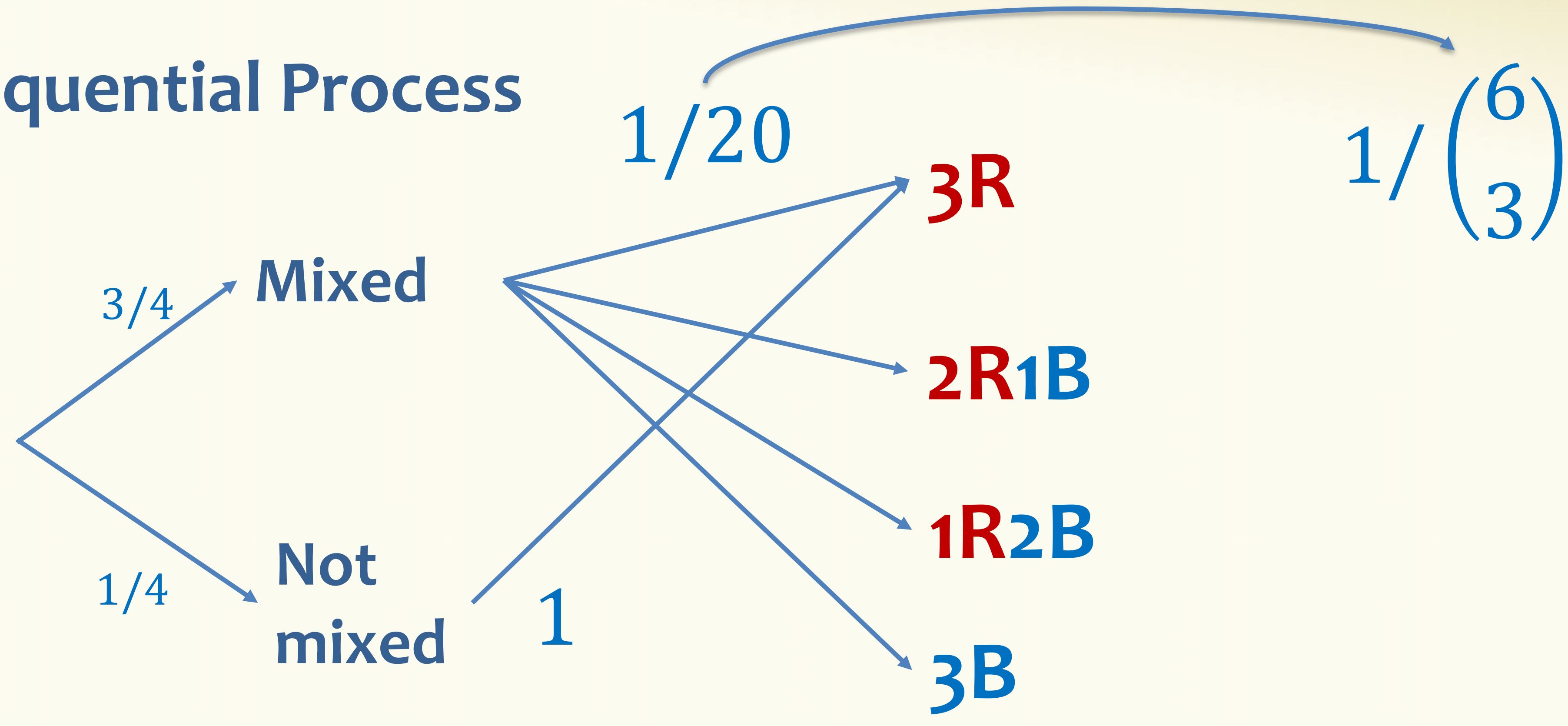
All three balls are red. What is the probability that the remaining (undrawn) balls are all blue?

Sequential Process



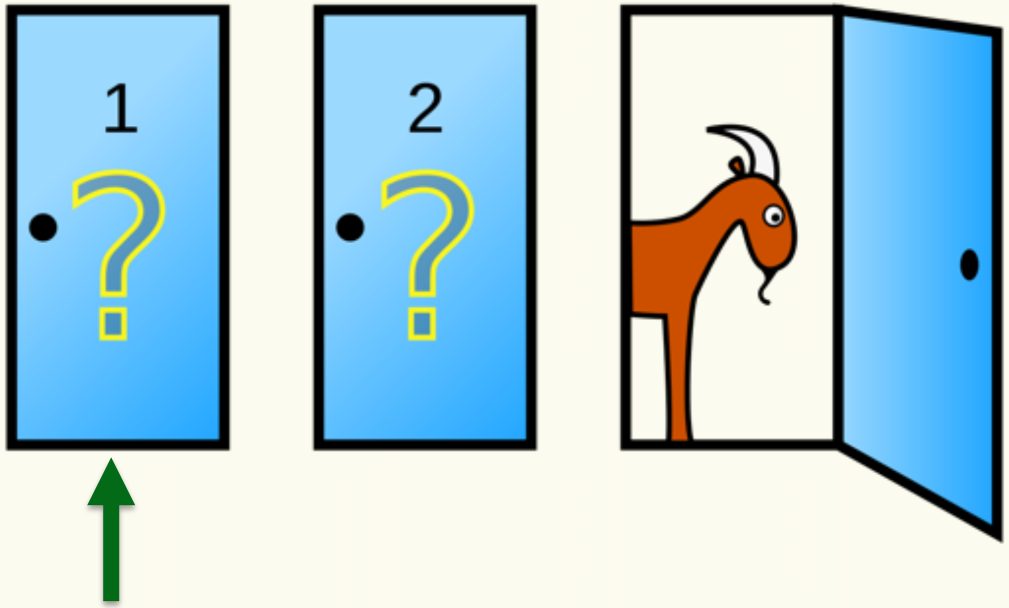
Wanted: $\mathbb{P}(\text{Mixed}|\text{3R})$

Sequential Process



$$\mathbb{P}(\text{Mixed}|\text{3R}) = \frac{\mathbb{P}(\text{Mixed})\mathbb{P}(\text{3R}|\text{Mixed})}{\mathbb{P}(\text{3R})} = \frac{\frac{3}{4}\times\frac{1}{20}}{\frac{3}{4}\times\frac{1}{20}+\frac{1}{4}\times 1} \approx 0.13$$

The Monty Hall Problem



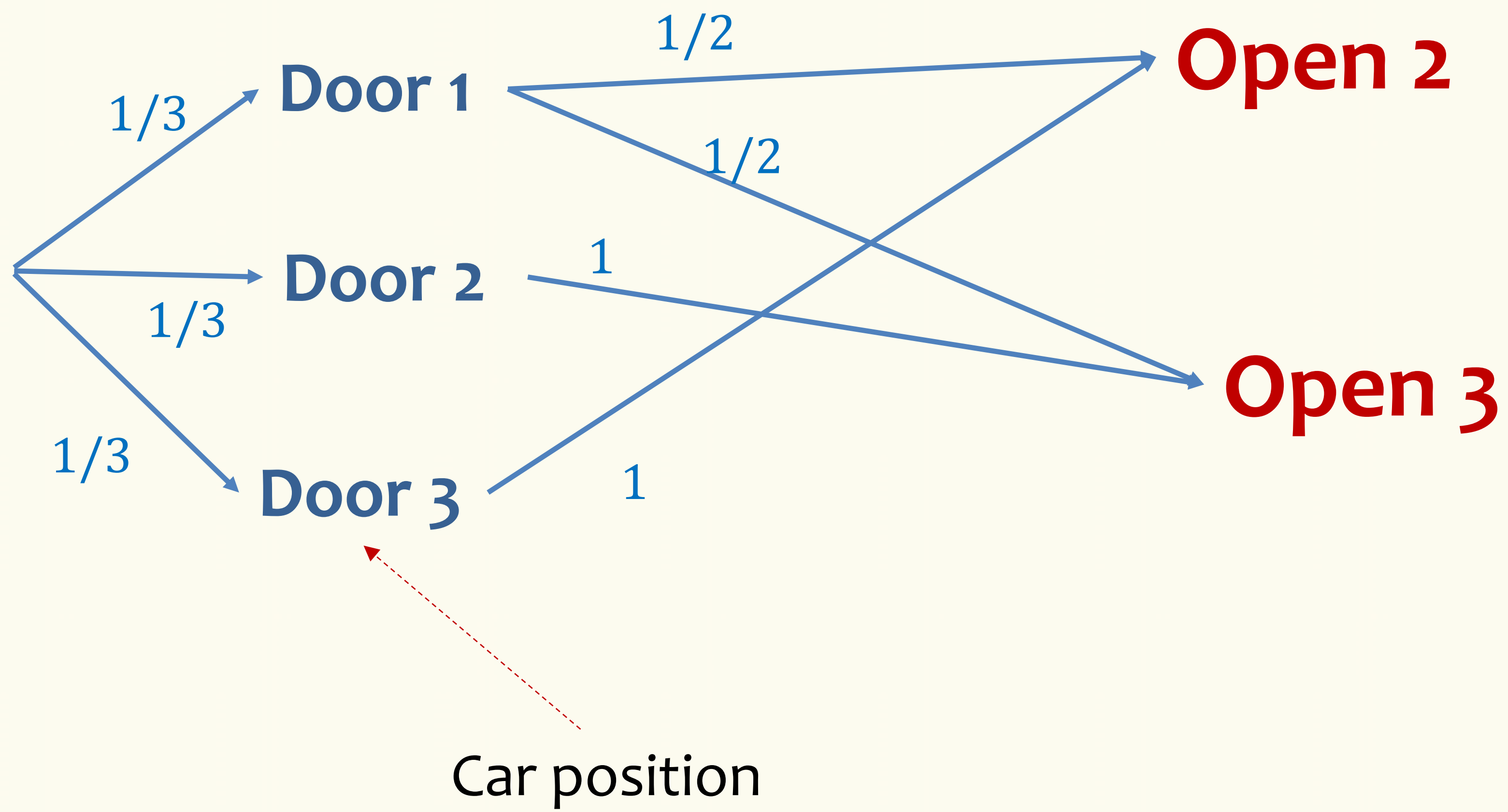
Your choice

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

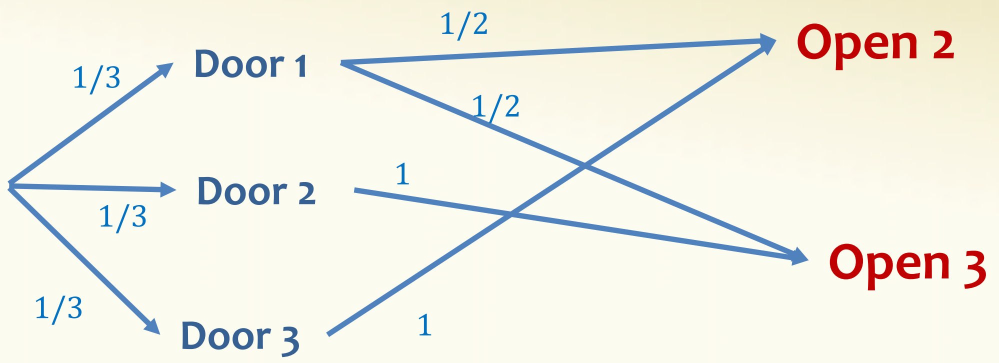
What would you do?

Monty Hall

Say you picked (without loss of generality) **Door 1**



Monty Hall



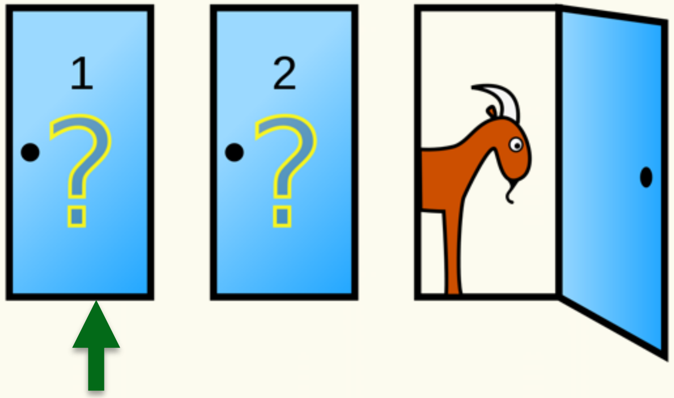
$$\mathbb{P}(\text{Door 1}|\text{Open 3}) = \frac{\mathbb{P}(\text{Door 1})\mathbb{P}(\text{Open 3}|\text{Door 1})}{\mathbb{P}(\text{Open 3})}$$

$$= \frac{\frac{1}{3}\times\frac{1}{2}}{\frac{1}{3}\times\frac{1}{2}+\frac{1}{3}\times 1} = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3}$$

$$\mathbb{P}(\text{Door 2}|\text{Open 3}) = 1 - \mathbb{P}(\text{Door 1}|\text{Open 3}) = 2/3$$

Monty Hall

Bottom line: Always swap!



Your
choice

7.2 – More on Independence

Independence – Recall

Definition. Two events $\mathcal{A}$ and $\mathcal{B}$ are (statistically) **independent** if

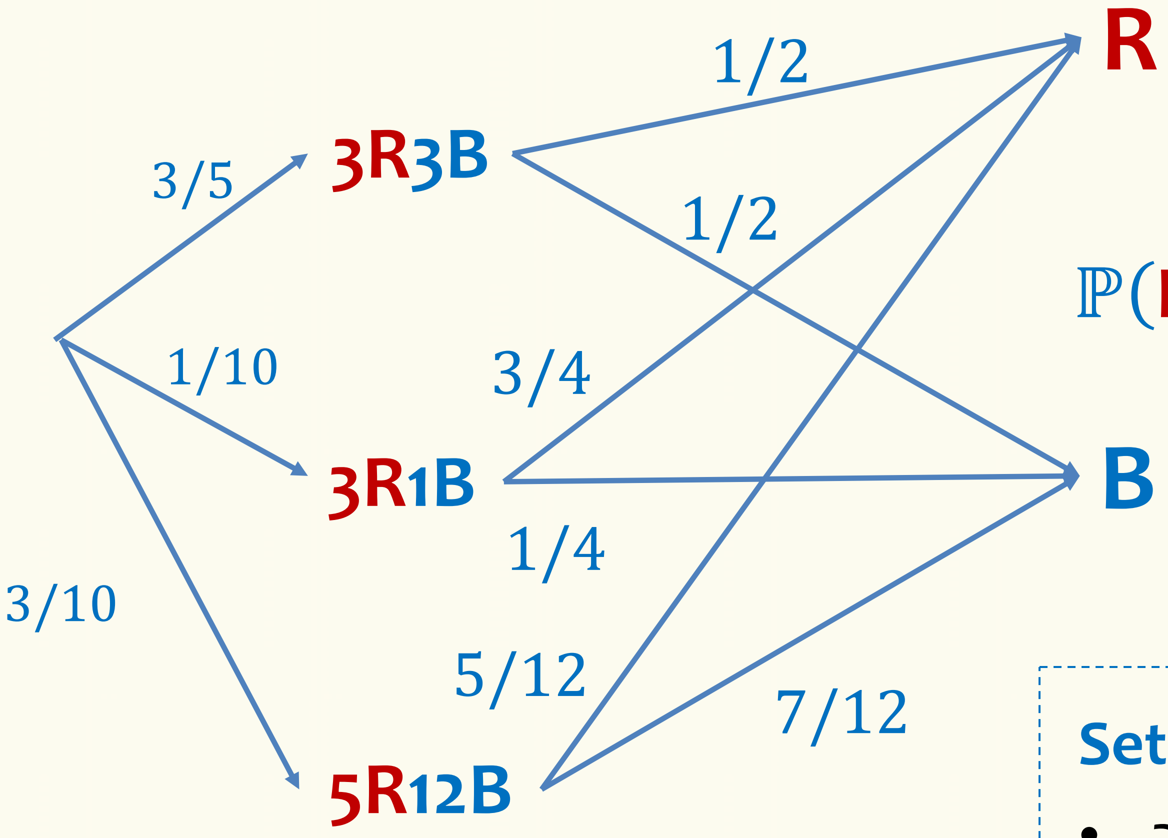
$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\mathcal{B}).$$

"Equivalently." $\mathbb{P}(\mathcal{A}|\mathcal{B}) = \mathbb{P}(\mathcal{A})$.

It is important to understand that independence is a property of probabilities of outcomes, not of the root cause generating these events.

This can be very counterintuitive!

Sequential Process



$$\mathbb{P}(\mathrm{R} \mid 3\mathrm{R}3\mathrm{B}) = \frac{1}{2}$$

$$\mathbb{P}(\mathrm{R}) = \frac{3}{5} \times \frac{1}{2} + \frac{1}{10} \times \frac{3}{4} + \frac{3}{10} \times \frac{5}{12} = \frac{1}{2}$$

Independent! $\mathbb{P}(\mathrm{R}) = \mathbb{P}(\mathrm{R} \mid 3\mathrm{R}3\mathrm{B})$

Are R and 3R3B independent?

Setting: An urn contains:

- 3 **red** and 3 **blue** balls w/ probability ¾
- 3 **red** and 1 **blue** balls w/ probability 1/10
- 5 **red** and 12 **blue** balls w/ probability 3/10

We draw a ball at random from the urn.

Independence – Multiple Events

Definition. Two events $\mathcal{A}$ and $\mathcal{B}$ are (statistically) **independent** if

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\mathcal{B}).$$

Equivalently. $\mathbb{P}(\mathcal{A}|\mathcal{B}) = \mathbb{P}(\mathcal{A})$.

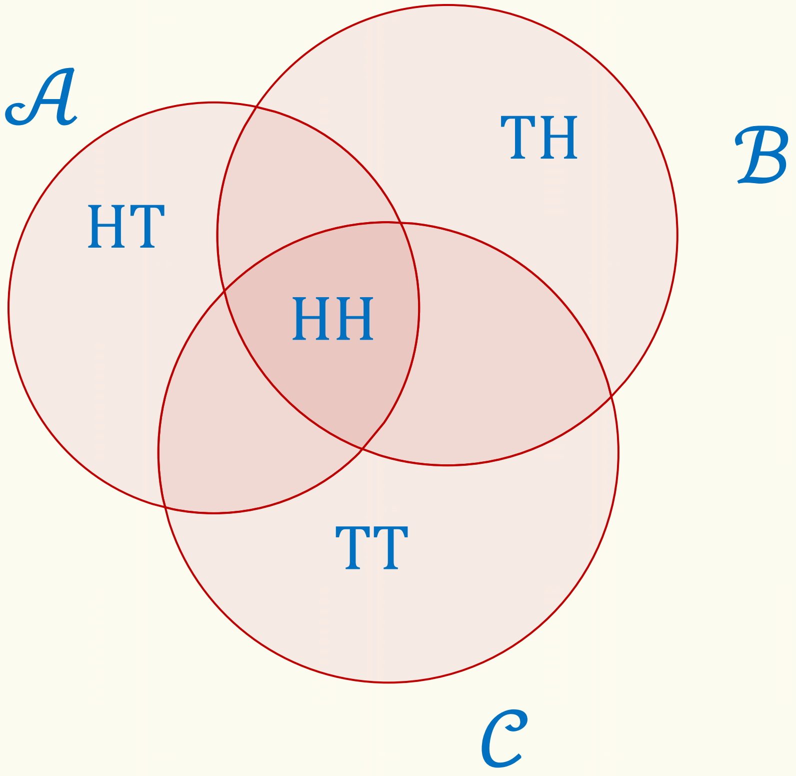
If we have more than two events, interesting phenomena can happen.

Example – Two Coin Tosses

"first coin is heads" $\mathcal{A} = \{\mathrm{HH}, \mathrm{HT}\}$ $\mathbb{P}(\mathcal{A}) = \frac{1}{2}$

"second coin is heads" $\mathcal{B} = \{\mathrm{HH}, \mathrm{TH}\}$ $\mathbb{P}(\mathcal{B}) = \frac{1}{2}$

"equal outcomes" $\mathcal{C} = \{\mathrm{HH}, \mathrm{TT}\}$ $\mathbb{P}(\mathcal{C}) = \frac{1}{2}$



$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\mathcal{B}) = \frac{1}{4}.$$

$$\mathbb{P}(\mathcal{A} \cap \mathcal{C}) = \mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\mathcal{C}) = \frac{1}{4}.$$

$$\mathbb{P}(\mathcal{B} \cap \mathcal{C}) = \mathbb{P}(\mathcal{B}) \cdot \mathbb{P}(\mathcal{C}) = \frac{1}{4}.$$

Every **pair** of events is independent

Pairwise Independence

Definition. The events $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are **pairwise-independent** if for all distinct $i, j \in [n]$,

$$\mathbb{P}\left(\mathcal{A}_i \cap \mathcal{A}_j\right) = \mathbb{P}(\mathcal{A}_i) \cdot \mathbb{P}(\mathcal{A}_j).$$

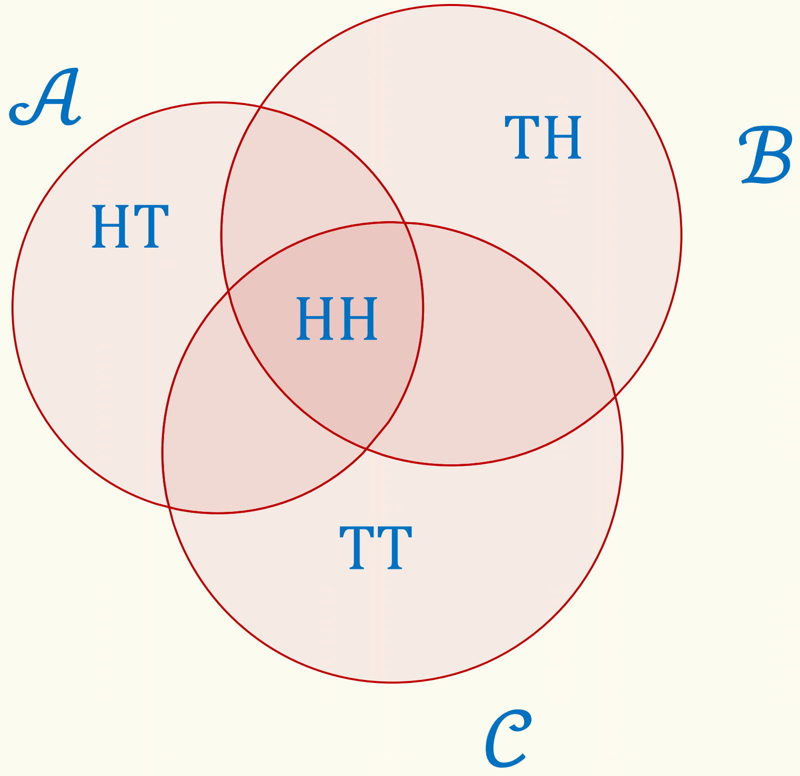
As we will see next week, pairwise independence is very powerful in computer science.

Example – Two Coin Tosses

"first coin is heads" $\mathcal{A} = \{\mathrm{HH}, \mathrm{HT}\}$ $\mathbb{P}(\mathcal{A}) = \frac{1}{2}$

"second coin is heads" $\mathcal{B} = \{\mathrm{HH}, \mathrm{TH}\}$ $\mathbb{P}(\mathcal{B}) = \frac{1}{2}$

"equal outcomes" $\mathcal{C} = \{\mathrm{HH}, \mathrm{TT}\}$ $\mathbb{P}(\mathcal{C}) = \frac{1}{2}$



$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\mathcal{B}) = \frac{1}{4}.$$

$$\mathbb{P}(\mathcal{A} \cap \mathcal{C}) = \mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\mathcal{C}) = \frac{1}{4}.$$

$$\mathbb{P}(\mathcal{B} \cap \mathcal{C}) = \mathbb{P}(\mathcal{B}) \cdot \mathbb{P}(\mathcal{C}) = \frac{1}{4}.$$

$\mathcal{A}, \mathcal{B}, \mathcal{C}$ are <u>pairwise</u> independent

Independence – Multiple Events

Definition. The events $\mathcal{A}_1, \dots, \mathcal{A}_n$ are **independent** if for every $k \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_k \leq n$,

$$\mathbb{P}\left(\mathcal{A}_{j_1} \cap \mathcal{A}_{j_2} \cap \cdots \cap \mathcal{A}_{j_k}\right) = \mathbb{P}\left(\mathcal{A}_{j_1}\right) \cdot \mathbb{P}\left(\mathcal{A}_{j_2}\right) \cdots \mathbb{P}\left(\mathcal{A}_{j_k}\right).$$

Fact. Pairwise independence does not imply independence!

Proof by counterexample*! (see next slide)

Fact. Independence implies pairwise-independence.

Trivial by definition, use $k = 2$

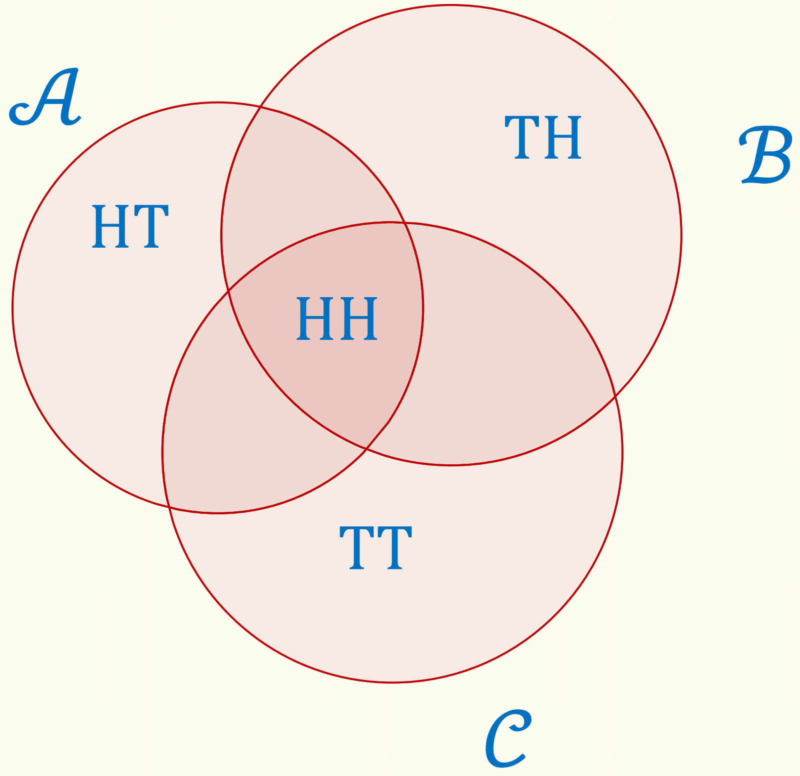
* Giving a counterexample is always sufficient to disprove an implication.

Example – Two Coin Tosses

"first coin is heads" $\mathcal{A} = \{\mathrm{HH}, \mathrm{HT}\}$ $\mathbb{P}(\mathcal{A}) = \frac{1}{2}$

"second coin is heads" $\mathcal{B} = \{\mathrm{HH}, \mathrm{TH}\}$ $\mathbb{P}(\mathcal{B}) = \frac{1}{2}$

"equal outcomes" $\mathcal{C} = \{\mathrm{HH}, \mathrm{TT}\}$ $\mathbb{P}(\mathcal{C}) = \frac{1}{2}$



$$\mathbb{P}(\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}) = \mathbb{P}(\mathrm{HH}) = \frac{1}{4}.$$

$$\frac{1}{4} \neq \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}.$$

$\mathcal{A}, \mathcal{B}, \mathcal{C}$ are <u>not</u> independent

Example – Two Coin Tosses

"first coin is heads"	$\mathcal{A} = \{\mathrm{HH}, \mathrm{HT}\}$	$\mathbb{P}(\mathcal{A}) = \frac{1}{2}$
"second coin is heads"	$\mathcal{B} = \{\mathrm{HH}, \mathrm{TH}\}$	$\mathbb{P}(\mathcal{B}) = \frac{1}{2}$
"equal outcomes"	$\mathcal{C} = \{\mathrm{HH}, \mathrm{TT}\}$	$\mathbb{P}(\mathcal{C}) = \frac{1}{2}$

$\mathcal{A}, \mathcal{B}, \mathcal{C}$ are <u>not</u> independent

Important: The formal notion matches the intuition, namely

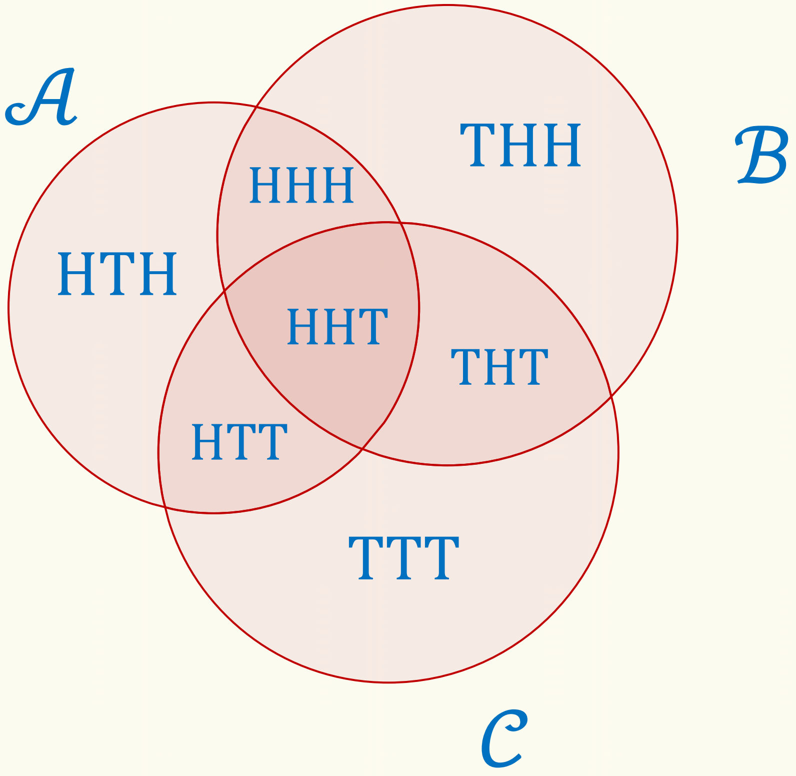
- If $\mathcal{A}$ and $\mathcal{B}$ have happened, we know both coins are heads.
- Therefore, $\mathcal{C}$ must have happened, i.e., $\mathbb{P}(\mathcal{C} | \mathcal{A} \cap \mathcal{B}) = 1$

Example – Three Coin Tosses

"first coin is heads" $\mathcal{A} = \{\text{HHH}, \text{HHT}, \text{HTH}, \text{HTT}\}$ $\mathbb{P}(\mathcal{A}) = \frac{1}{2}$

"second coin is heads" $\mathcal{B} = \{\text{HHH}, \text{HHT}, \text{THH}, \text{THT}\}$ $\mathbb{P}(\mathcal{B}) = \frac{1}{2}$

"third coin is tails" $\mathcal{C} = \{\text{HHT}, \text{HTT}, \text{THT}, \text{TTT}\}$ $\mathbb{P}(\mathcal{C}) = \frac{1}{2}$



$$\mathbb{P}(\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}) = \mathbb{P}(\text{HHT}) = \frac{1}{8}.$$

$$= \mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\mathcal{B}) \cdot \mathbb{P}(\mathcal{C})$$

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = \mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\mathcal{B})$$

Similarly: $\mathbb{P}(\mathcal{A} \cap \mathcal{C}) = \mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\mathcal{C})$

$\mathbb{P}(\mathcal{B} \cap \mathcal{C}) = \mathbb{P}(\mathcal{B}) \cdot \mathbb{P}(\mathcal{C})$

$\rightarrow \mathcal{A}, \mathcal{B}, \mathcal{C}$ are independent

Independence & Conditioning

Conditioning can break independence.

"first coin is heads"	$\mathcal{A} = \{\mathrm{HH}, \mathrm{HT}\}$	$\mathbb{P}(\mathcal{A}) = \frac{1}{2}$
"second coin is tails"	$\mathcal{B} = \{\mathrm{HT}, \mathrm{TT}\}$	$\mathbb{P}(\mathcal{B}) = \frac{1}{2}$
"equal outcomes"	$\mathcal{C} = \{\mathrm{HH}, \mathrm{TT}\}$	$\mathbb{P}(\mathcal{C}) = \frac{1}{2}$

$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\mathcal{B}) = \frac{1}{4}$. $\mathbb{P}(\mathcal{A} \cap \mathcal{B} | \mathcal{C}) = \mathbb{P}(\mathcal{A} | \mathcal{C}) \cdot \mathbb{P}(\mathcal{B} | \mathcal{C})$?

$\mathbb{P}(\mathcal{A} \cap \mathcal{B} | \mathcal{C}) = 0$ b/c if both outcomes are equal, we cannot have $\mathcal{A} \cap \mathcal{B}$

$$\mathbb{P}(\mathcal{A} | \mathcal{C}) = \frac{\mathbb{P}(\mathcal{A} \cap \mathcal{C})}{\mathbb{P}(\mathcal{C})} = \frac{\mathbb{P}(\mathrm{HH})}{\mathbb{P}(\mathcal{C})} = \frac{1}{4} \times \frac{2}{1} = \frac{1}{2} \qquad \mathbb{P}(\mathcal{B} | \mathcal{C}) = \frac{1}{2}$$