CSE 312

Foundations of Computing II

Lecture 4: Inclusion-exclusion principle

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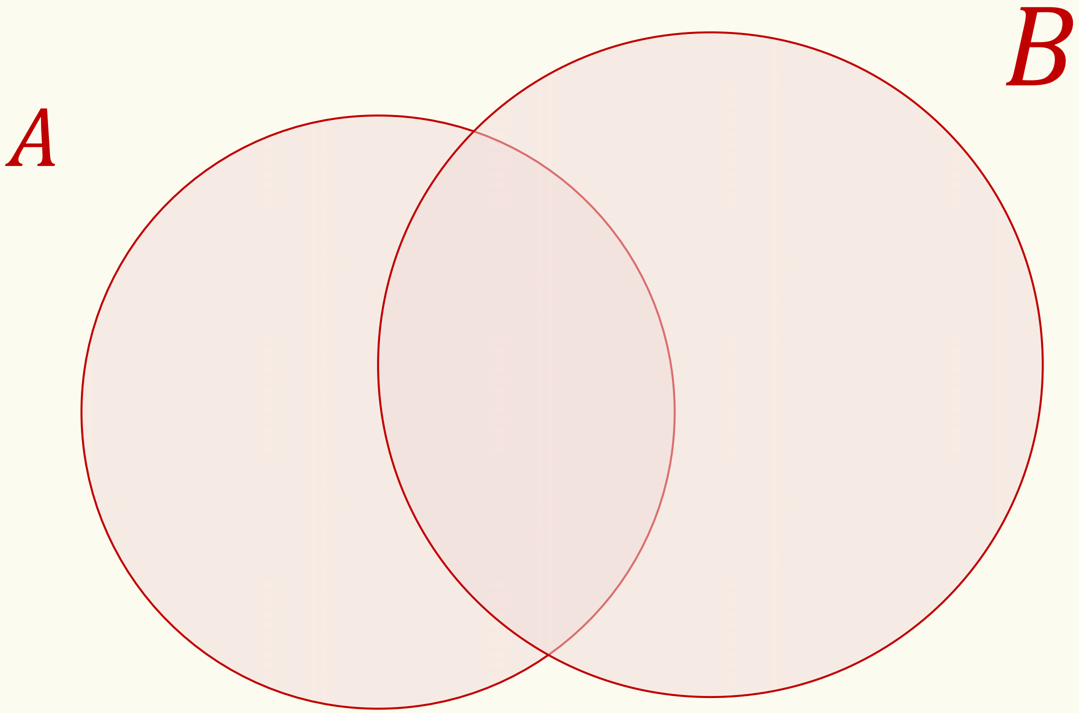
Announcements

- Homework online tonight by 11:59pm.
- Go to sections tomorrow.

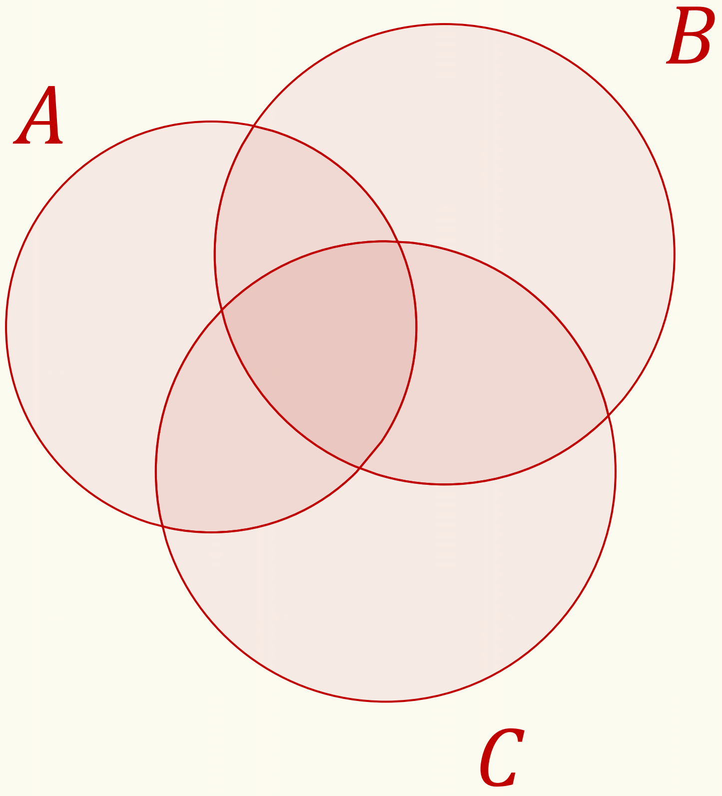
Inclusion-Exclusion

Sometimes, we want $|S|$, and $S = A \cup B$



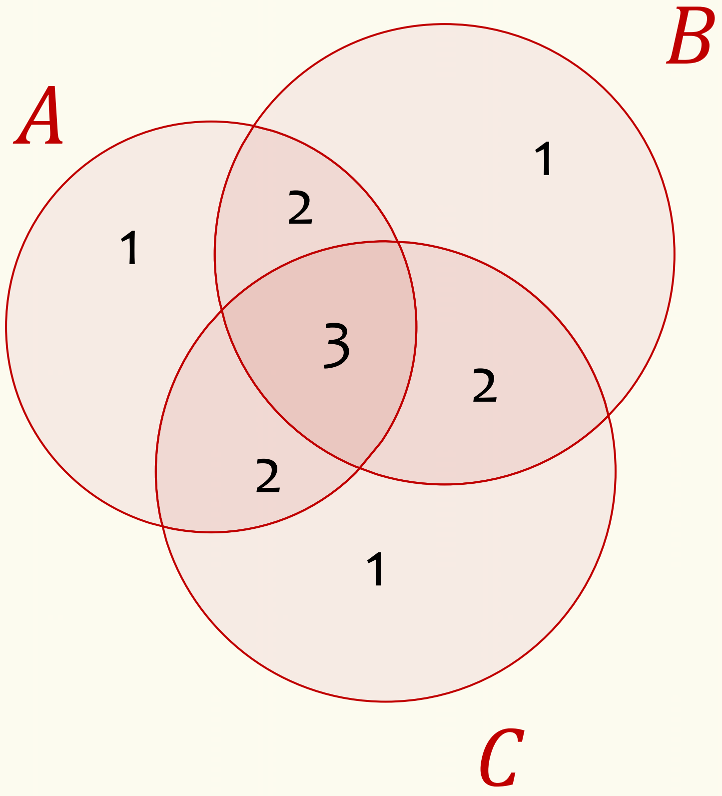
Fact. $|A \cup B| = |A| + |B| - |A \cap B|$

Inclusion-Exclusion – Three Sets



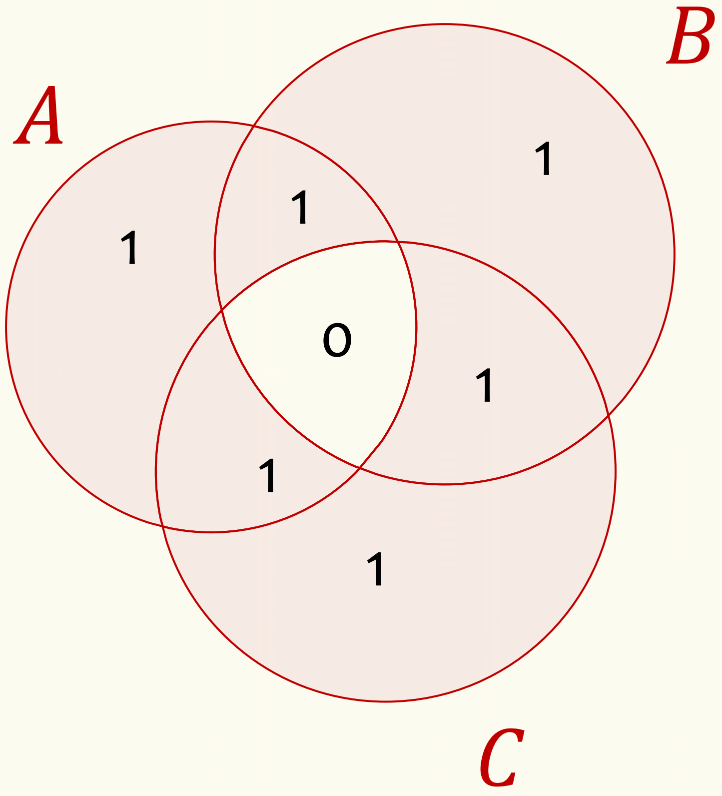
$$|A \cup B \cup C|?$$

Inclusion-Exclusion – Three Sets



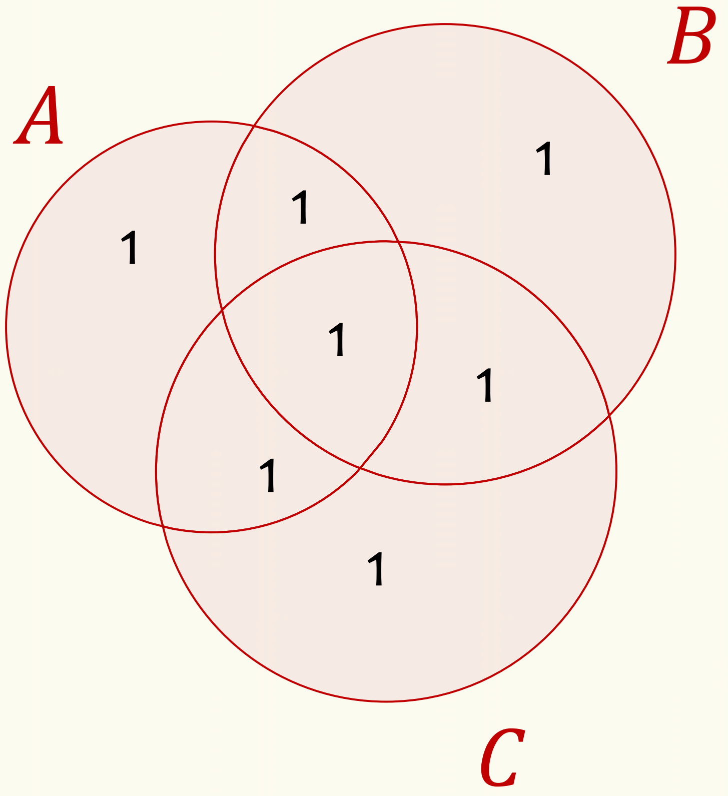
$$|A| + |B| + |\mathrm{C}|$$
$$-|A \cap B| - |A \cap C| - |B \cap C|$$

Inclusion-Exclusion – Three Sets



$$|A| + |B| + |C|$$
$$-|A \cap B| - |A \cap C| - |B \cap C|$$
$$+|A \cap B \cap C|$$

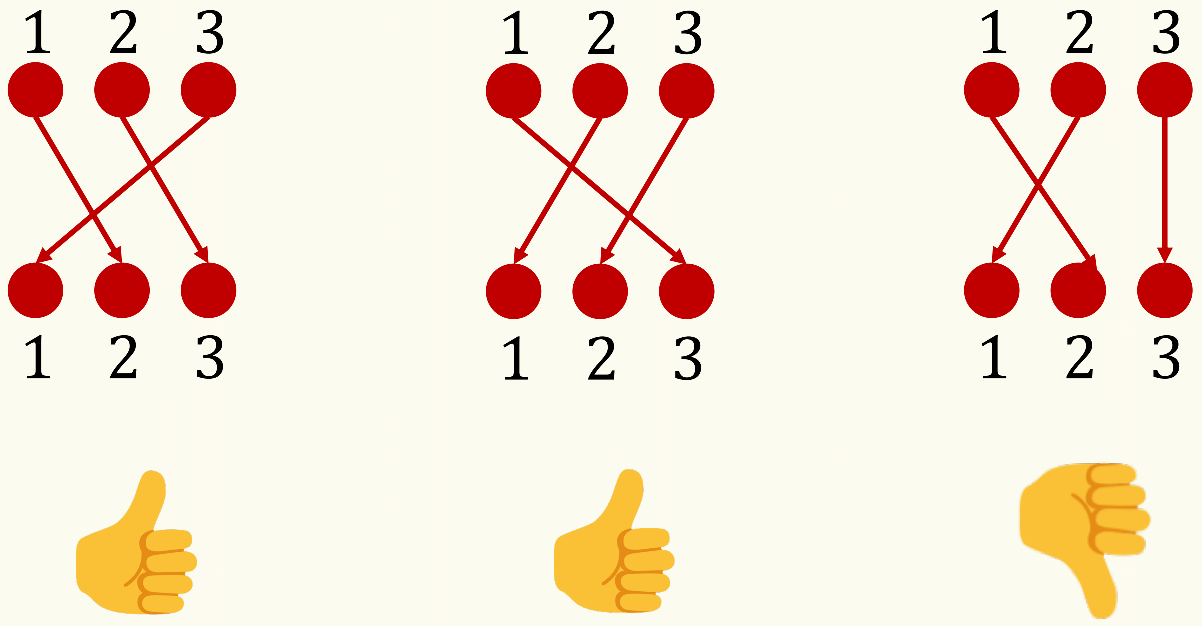
Inclusion-Exclusion – Three Sets



$$|A \cup B \cup C| =$$
$$|A| + |B| + |C|$$
$$-|A \cap B| - |A \cap C| - |B \cap C|$$
$$+|A \cap B \cap C|$$

Example – Number of Derangements

How many one-to-one maps $\pi: [3] \to [3]$ *are there such that* $\pi(i) \neq i$ *for all* i*?*



Example – Number of Derangements

How many one-to-one maps $\pi: [3] \rightarrow [3]$ *are there such that* $\pi(i) \neq i$ *for all* i*?*

Alternatively:

In how many ways can we arrange 3 *people such that none of them stays in place?*

In how many ways can we have students grade each other's homework without anyone grading their own homework?

Example – Number of Derangements

How many one-to-one maps $\pi\colon [3] \to [3]$ *are there such that* $\pi(i) \neq i$ *for all* i*?*

$S_3 =$ all one-to-one $\pi\colon [3] \to [3]$

$A =$ all $\pi \in S_3$ s.t. $\pi(1) = 1$

$B =$ all $\pi \in S_3$ s.t. $\pi(2) = 2$

$C =$ all $\pi \in S_3$ s.t. $\pi(3) = 3$

Wanted: $|S_3 \setminus (A \cup B \cup C)| = \underbrace{|S_3|}_{= 3!} - \underbrace{|A \cup B \cup C|}_{=?}$

Example – Number of Derangements

How many one-to-one maps $\pi: [3] \to [3]$ *are there such that* $\pi(i) \neq i$ *for all* i*?*

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$C =$ all $\pi \in S_3$ s.t. $\pi(3) = 3$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$= 2! = 2$ $\qquad$ $= 1! = 1$ $\qquad$ $= 0! = 1$

Example – Number of Derangements

How many one-to-one maps π: $[3] \rightarrow [3]$ *are there such that* $\pi(i) \neq i$ *for all* i*?*

$S_3 =$ all π: $[3] \rightarrow [3]$

$A =$ all $\pi \in S_3$ s.t. $\pi(1) = 1$

$B =$ all $\pi \in S_3$ s.t. $\pi(2) = 2$

$C =$ all $\pi \in S_3$ s.t. $\pi(3) = 3$

$|A \cup B \cup C| = 3\times 2 - 3\times 1 + 1 = 4$

$|S_3 \setminus (A \cup B \cup C)| = |S_3| - |A \cup B \cup C| = 3! - 4 = 2$

General Case – Number of Derangements

How many one-to-one maps $\pi: [n] \rightarrow [n]$ *are there such that* $\pi(i) \neq i$ *for all* $i \in [n]$*?*

We have seen that 1/3 permutations over $[3]$ are derangements

Any guesses for the general case? Vanishing fraction? Constant fraction?

Inclusion-exclusion – General formula

Theorem. For any nv(finite) sets $A_1, \dots, A_n$,

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|+1} \left|\bigcap_{i \in I} A_i\right|$$

sum over all subsets of $[n]$, <u>except</u> the empty set

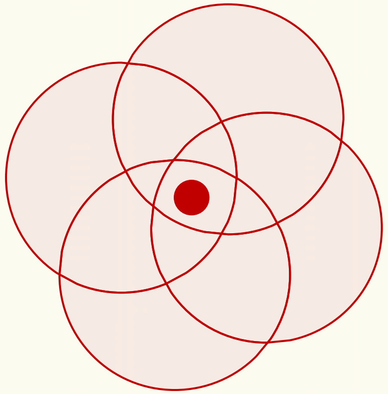
+ sign for odd-sized sets,
- sign for even-sized sets

Inclusion-exclusion – Proof (sketch)

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|+1} \left|\bigcap_{i \in I} A_i\right|$$

Need to verify every element $x \in \bigcup_{i=1}^{n} A_i$ is counted exactly <u>once</u>.

Assume x is contained in $1 \leq k \leq n$ sets – call these $A_{i_1}, \ldots, A_{i_k}$



In formula, x is counted

$$k - \binom{k}{2} + \binom{k}{3} - \binom{k}{4} + \cdots = \sum_{i\ odd} \binom{k}{i} - \sum_{i\ even, i>0} \binom{k}{i} = 1$$

Why? Binomial theorem $\Longrightarrow 0 = (1-1)^k = \sum_{i=0}^{k} (-1)^i \binom{k}{i}$

Example – Number of Derangements

How many one-to-one maps $\pi: [n] \to [n]$ *are there such that* $\pi(i) \neq i$ *for all* $i \in [n]$*?*

S_n = all one-to-one $\pi: [3] \to [3]$

A_i = all $\pi \in S_n$ s.t. $\pi(i) = i$

Fact. $\left|\bigcap_{i \in I} A_i\right| = (n - |I|)!$

Wanted:

$$|S_n \setminus (A_1 \cup A_2 \cup \cdots \cup A_n)| = \underbrace{|S_n|}_{= n!} - \underbrace{|A_1 \cup A_2 \cup \cdots \cup A_n|}_{=?}$$

Fact. $\left|\bigcap_{i\in I} A_i\right| = (n-|I|)!$

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\emptyset\neq I\subseteq[n]} (-1)^{|I|+1}\left|\bigcap_{i\in I} A_i\right|$$

$$= \sum_{\emptyset\neq I\subseteq[n]} (-1)^{|I|+1}(n-|I|)!$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$= \sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}(n-k)! = n!\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k!} = -n!\sum_{k=1}^{n}\frac{(-1)^{k}}{k!}$$

$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}$$

$$\left|S_n \setminus \left(\bigcup_{i=1}^{n} A_i\right)\right| = n! - \left|\bigcup_{i=1}^{n} A_i\right| = n! + n!\sum_{k=1}^{n}\frac{(-1)^{k}}{k!} = n!\sum_{k=0}^{n}\frac{(-1)^{k}}{k!} \to \frac{n!}{e}$$

[Indeed: result is integer closest to $\frac{n!}{e}$]

Back to Euler's Totient Function

Definition. The **Euler totient function** is defined as

$$\varphi(N) = |\{a \in [N] \mid \gcd(a, N) = 1\}|$$

Goal: Give a formula for $\varphi(N)$.

Assume $N = P_1^{e_1} P_2^{e_2} \dots P_k^{e_k}$ where $P_1, \dots, P_k$ are distinct primes (by the fundamental theorem of arithmetic, this factorization is <u>unique</u>).

$A_i =$ multiples of P_i in $[N]$

$\Rightarrow |A_i| = N/P_i$

Fact. $\left|\bigcap_{i \in I} A_i\right| = \frac{N}{\prod_{i \in I} P_i}$

We know (see last time):

$$\varphi(N) = \left|[N] \setminus \left(\bigcup_{i=1}^{k} A_i\right)\right| = N - \left|\bigcup_{i=1}^{k} A_i\right|$$

Fact. $\left|\bigcap_{i\in I} A_i\right| = \frac{N}{\prod_{i\in I} P_i}$

$N = P_1^{e_1} P_2^{e_2} \dots P_k^{e_k}$

$$\varphi(N) = N - \left|\bigcup_{i=1}^{k} A_i\right| = N - \sum_{\emptyset\neq I\subseteq[n]} (-1)^{|I|+1} \left|\bigcap_{i\in I} A_i\right|$$

$$= N + \sum_{\emptyset\neq I\subseteq[n]} (-1)^{|I|} \frac{N}{\prod_{i\in I} P_i}$$

$$= N \sum_{I\subseteq[n]} (-1)^{|I|} \frac{1}{\prod_{i\in I} P_i} = N \sum_{I\subseteq[n]} (-1)^{|I|} \prod_{i\in I} \frac{1}{P_i}$$

$$= N \sum_{I\subseteq[n]} \prod_{i\in I} \left(-\frac{1}{P_i}\right)$$

e.g. $\left(1-\frac{1}{P_1}\right)\left(1-\frac{1}{P_2}\right) = 1 - \frac{1}{P_1} - \frac{1}{P_2} + \frac{1}{P_1 P_2}$

$$= N \prod_{i=1}^{k} \left(1 - \frac{1}{P_i}\right) = \prod_{i=1}^{k} P_i^{e_i - 1}(P_i - 1)$$