CSE 312

Foundations of Computing II

Lecture 27: Randomized Algorithms I

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Announcements

- HW8 due Friday, <u>no extensions</u>
- Final instructions have been posted
- Practice finals have been posted
  - Discussed in Sections on Thursday
- Review section on Wednesday
- Please complete the class evaluation!

Algorithms can be randomized

```java
Random rand = new Random();
int value = rand.nextInt(50);
```

Of course, not <u>really</u> random. But outcome can be approximate well by appropriate random variable.

```java
double random = Math.random() * 49 + 1;
```

```java
Random rand = new SecureRandom();
int value = rand.nextInt(50);
```

As an aside: For strong cryptographic random generator, finding non-random behavior would be a breakthrough

Randomized Algorithms – Two types

- **Las Vegas:** Guaranteed correct output
  - Running time is a random variable $T(n)$.
  - Complexity measured in terms of $\mathbb{E}\left(T(n)\right)$

- **Monte Carlo:** Guaranteed running time
  - Output is a random variable
  - Can make errors
  - Quality measured in terms of error probability

Quicksort – Recap

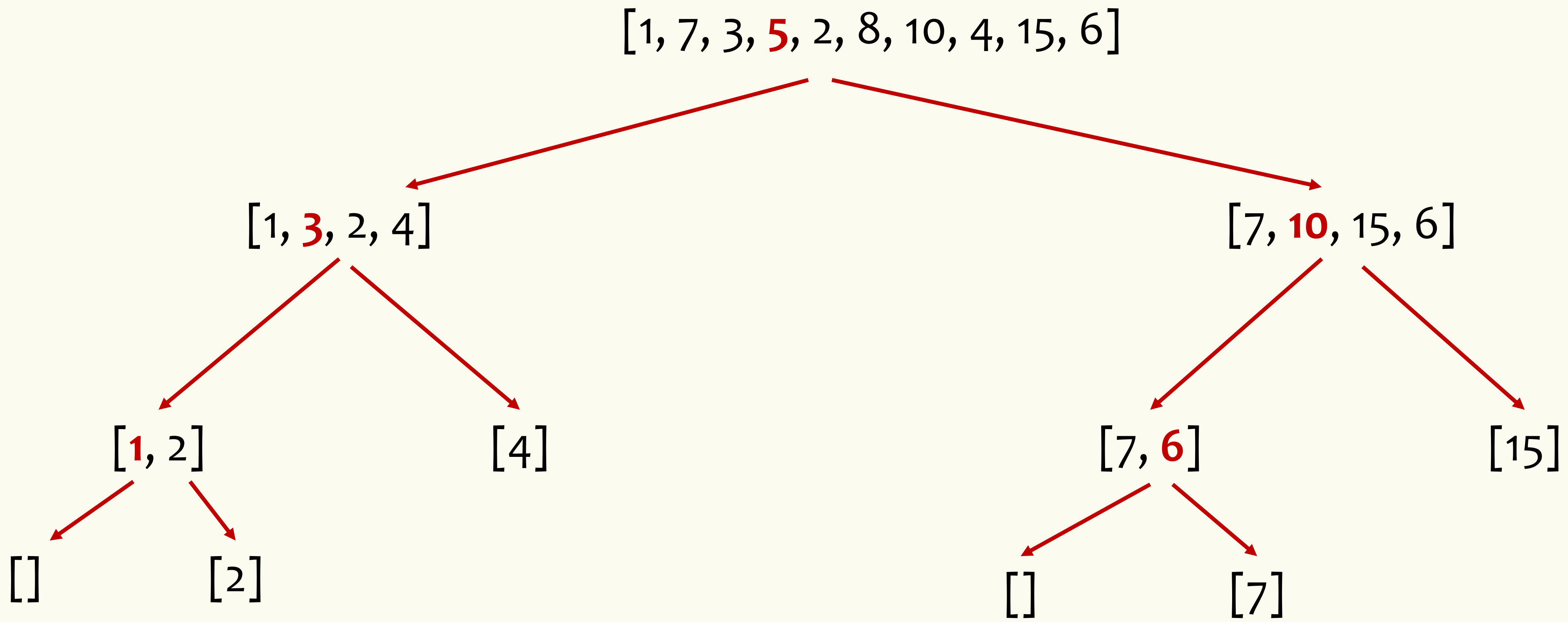
(Assume for simplicity no repeated elements)

Algorithm QuickSort(A): // A is array of size n

1) If $n \in \{0,1\}$ then **return** A
2) Choose pivot p from A
3) Let $A_0 =$ elements of A which are $< p$
4) Let $A_1 =$ elements of A which are $> p$
5) **Return** QuickSort(A_0) || p || QuickSort(A_1)

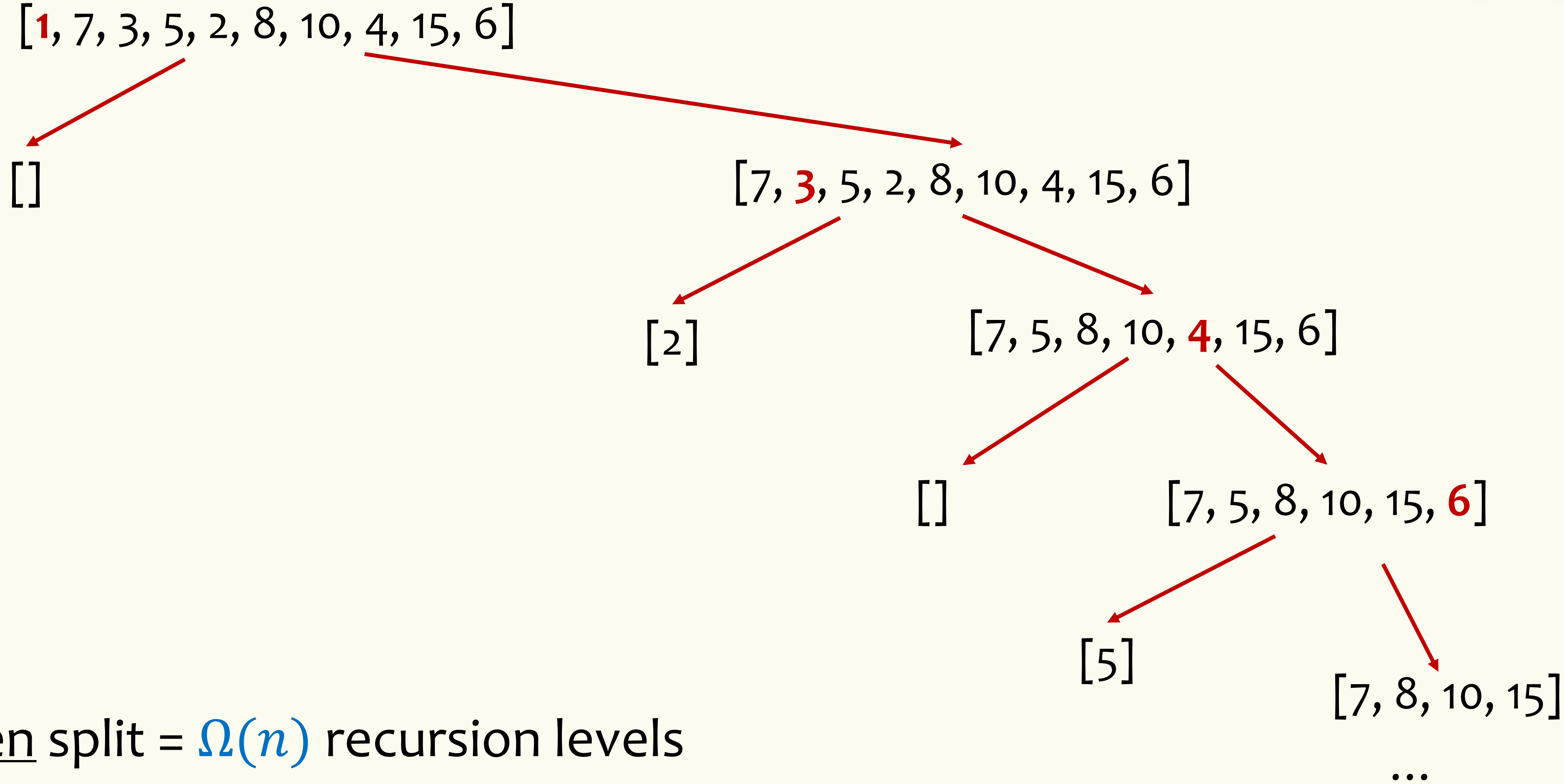
Can be done with $n - 1$ comparisons

Recursion Tree – Good Pivot



even split = $O(\log n)$ recursion levels

Recursion Tree – Bad Pivot



<u>uneven</u> split = $\Omega(n)$ recursion levels

A Las Vegas Algorithm – Randomized Quicksort

Algorithm `QuickSort`(A): // A is array of size n

1) If $n \in \{0,1\}$ then **return** A
2) Pivot p – **random** element from A
3) Let $A_0 =$ elements of A which are $< p$
4) Let $A_1 =$ elements of A which are $> p$

Can be done with $n-1$ comparisons

5) **Return** `QuickSort`(A_0) || p || `QuickSort`(A_1)

Goal – Count comparisons

- $T(n) =$ # of comparisons on input n-element array
  - **Goal:** Compute $\mathbb{E}(T(n))$ (Approximation of expected runtime)
- $e_1 < e_2 < \cdots < e_n$ are distinct elements of the array (when sorted)
  - $X_{ij} = 1$ if element $e_i < e_j$ are ever compared, 0 else
  - Two elements can be compared at most once (one of them must be a pivot)!
  - Therefore: $\mathbb{E}(T(n)) = \mathbb{E}\left(\sum_{i<j} X_{ij}\right) = \sum_{i<j} \mathbb{E}\left(X_{ij}\right) = \sum_{i<j} \mathbb{P}\left(X_{ij} = 1\right)$

Example: $e_i = \mathbf{3}$, $e_j = \mathbf{7}$

[1, 7, 3, 5, 2, 8, 10, 4, 15, 6]

[1, 3, 2, 4]

[7, 10, 15, 6]

Never compared, because first pivot **5** separates them into two different sub-arrays by being between **3** and **7**.

Therefore: $X_{ij} = 0$

Example: $e_i = \mathbf{7}$, $e_j = \mathbf{10}$

[1, **7**, 3, **5**, 2, 8, **10**, 4, 15, 6]

[1, 3, 2, 4]   [**7**, **10**, 15, 6]

Compared, because on the same side for pivot **5**, then one of them is chosen as pivot.

Therefore: $X_{ij} = 1$

Summarizing

Recall: $e_1 < e_2 < \cdots < e_n$ are distinct elements of the array

X_{ij} is determined by following process:

- Pick (random) pivot p
- If $p \in [e_i, e_j]$ then
  - If $p = e_i$ or $p = e_j$ then $X_{ij} = 1$
  - If $p \neq e_i, e_j$ then $X_{ij} = 0$
- Else try another round

$\mathcal{A}_k$ = X_{ij} is set after exactly k iterations

$$\mathbb{P}(X_{ij} = 1 | \mathcal{A}_k) = \frac{2}{j - i + 1}$$

$$\mathbb{P}\left(X_{ij} = 1\right) = \sum_k \mathbb{P}(\mathcal{A}_k) \cdot \mathbb{P}(X_{ij} = 1 | \mathcal{A}_k) = \frac{2}{j - i + 1} \sum_k \mathbb{P}(\mathcal{A}_k) = \frac{2}{j - i + 1}$$

Randomized Quicksort – Wrapping up

$$\mathbb{P}\left(X_{ij}=1\right)=\sum_{k}\mathbb{P}(\mathcal{A}_k)\cdot\mathbb{P}(X_{ij}=1|\mathcal{A}_k)=\frac{2}{j-i+1}\sum_{k}\mathbb{P}(\mathcal{A}_k)=\frac{2}{j-i+1}$$

$$\mathbb{E}(T(n))=\sum_{i<j}\mathbb{P}\left(X_{ij}=1\right)=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{2}{j-i+1}$$

$$=\sum_{i=1}^{n-1}\sum_{j=2}^{n-i+1}\frac{2}{j}\leq 2\sum_{i=1}^{n-1}\sum_{j=1}^{n}\frac{1}{j}=2\sum_{i=1}^{n-1}H_n\leq 2nH_n\sim 2n\ln n$$

Monte Carlo Algorithms – Primality Testing

Question: Is an integer N prime?

- Is 7 prime?
- Is 25 prime?
- Is 23 prime?
- Is 7919 prime?
  - Yes! 1000^{th} prime!
- Is $1230186684530117755130494958384962720772853569595334$
$792197322452151726400507263657518745202199786469389956474942 7$
$74063845925192557326303453731548268507917026122142913461670 42$
$9214311602221240479274737794080665351419597459856902143413$
prime?
  - No ;) [It’s the product of two large primes, very hard to factor!]

Primality – Deterministic Complexity

- Trivial algorithm runs in time (roughly) $O(N \log N)$
  - Check divisibility by every integer $1 < i < N$
  - Can be optimized to $O(\sqrt{N} \log N)$ [Why?]
- **Breakthrough result (Agrawal–Kayal–Saxena, 2006):** Primality testing in $O((\log N)^{7.5})$
  - Much better, but still not very practical …
- Testing primality is very useful in cryptography (and elsewhere)

Note: Deciding whether an integer is prime or not is much easier than finding the prime factors if it is not prime!

Primality – The Miller-Rabin Test

Running time can be made (almost) $O(\log(N)^2)$

Algorithm `IsPrime`(N):

- $s = 0, d = N - 1$
- **while** d is even **do**
  - $d = \frac{d}{2}; \quad s = s + 1$
- Pick uniformly $b \in \{1,2,\dots,N-1\}$
- $t = b^d$
- **For** $i = 1$ to s **do**
  - If $t = N - 1$ **then return** 👍
  - $t = t^2 \bmod N$
- **Return** 👎

At the end of loop: $N - 1 = 2^s d$

Checks whether $N - 1$ is any of $b^d, b^{2d}, b^{4d}, \dots, b^{2^{s-1}d} \bmod N$

Miller-Rabin Primality Test

Theorem. The Miller-Rabin test satisfies the following properties:

- If N is prime, then $\mathbb{P}(\texttt{IsPrime}(N)) = 1$
- If N is not prime, then $\mathbb{P}(\texttt{IsPrime}(N)) \leq 1/4$

For better guarantees:

- Repeat k times, return prime only if all tests return prime.
- If N is not prime, prob. of incorrectly identifying it as prime is $\leq 4^{-k} = 2^{-2k}$
  - E.g., $k = 64$, we have 2^{-128}