

CSE 312

Foundations of Computing II

Lecture 21: The Normal Distribution



Stefano Tessaro

tessaro@cs.washington.edu

The Normal Distribution

Definition. A **Gaussian (or normal) random variable** with parameters $\mu \in \mathbb{R}$ and $\sigma^2 \geq 0$ has density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

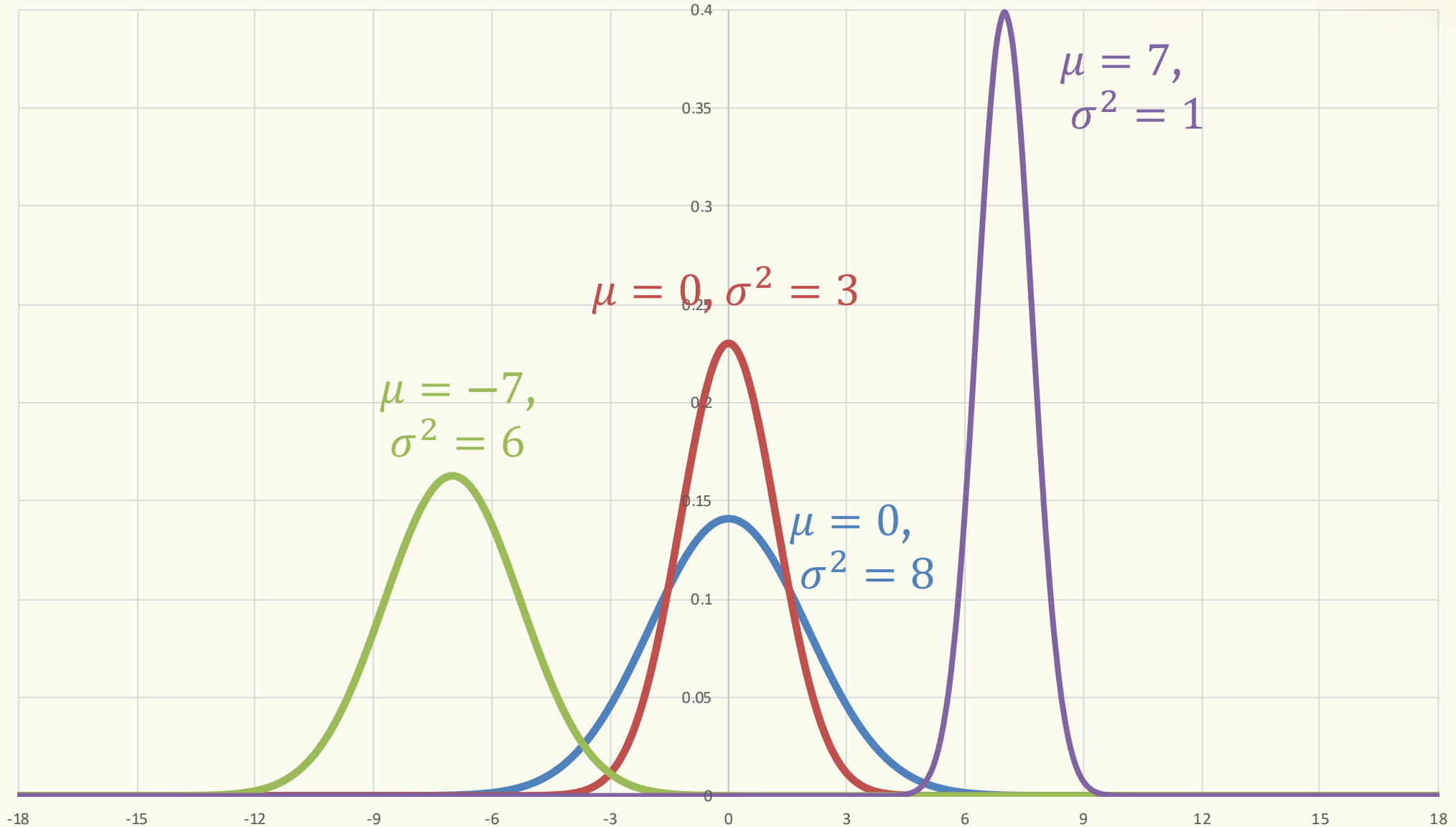
We say that X follows the Normal Distribution, and write $X \sim \mathcal{N}(\mu, \sigma^2)$



Carl Friedrich
Gauss

The Normal Distribution

Aka a “Bell Curve” (imprecise name)



Important Facts

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\mathbb{E}(X) = \mu$

Proof is easy b/c density is symmetric around μ , and $f_X(\mu - x) = f_X(\mu + x)$

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\text{Var}(X) = \sigma^2$

Proof requires integration, see textbook

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

How do we prove this? (Likely) see next week for a simple proof.

Shifting and Scaling

Standard (unit) normal = $\mathcal{N}(0, 1)$

Definition. $\Phi(z) = \mathbb{P}(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-x^2/2} dx$ for $Z \sim \mathcal{N}(0, 1)$

Note: $\Phi(z)$ has no closed form – generally given via tables

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$

Recall: $aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

$$a = 1/\sigma \quad b = -\mu/\sigma$$

$$F_X(z) = \mathbb{P}(X \leq z) = \mathbb{P}\left(\frac{X - \mu}{\sigma} \leq \frac{z - \mu}{\sigma}\right) = \Phi\left(\frac{z - \mu}{\sigma}\right)$$

Table of Standard Cumulative Normal Density

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451

Example

Let $X \sim \mathcal{N}(0.4, 4 = 2^2)$.

$$\begin{aligned}\mathbb{P}(X \leq 1.2) &= \mathbb{P}\left(\frac{X - 0.4}{2} \leq \frac{1.2 - 0.4}{2}\right) \\ &= \mathbb{P}\left(\frac{X - 0.4}{2} \leq 0.4\right) = \Phi(0.4) \approx 0.6554\end{aligned}$$

$\sim \mathcal{N}(0, 1)$

0.1	0.5398	0.5438
0.2	0.5793	0.5832
0.3	0.6179	0.6217
0.4	0.6554	0.6591
0.5	0.6915	0.6950
0.6	0.7257	0.7291
0.7	0.7580	0.7611

Example – Off by Standard Deviations

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

$$\begin{aligned}\mathbb{P}(|X - \mu| < k\sigma) &= \mathbb{P}\left(\frac{|X - \mu|}{\sigma} < k\right) = \\ &= \mathbb{P}\left(-k < \frac{X - \mu}{\sigma} < k\right) = \Phi(k) - \Phi(-k)\end{aligned}$$

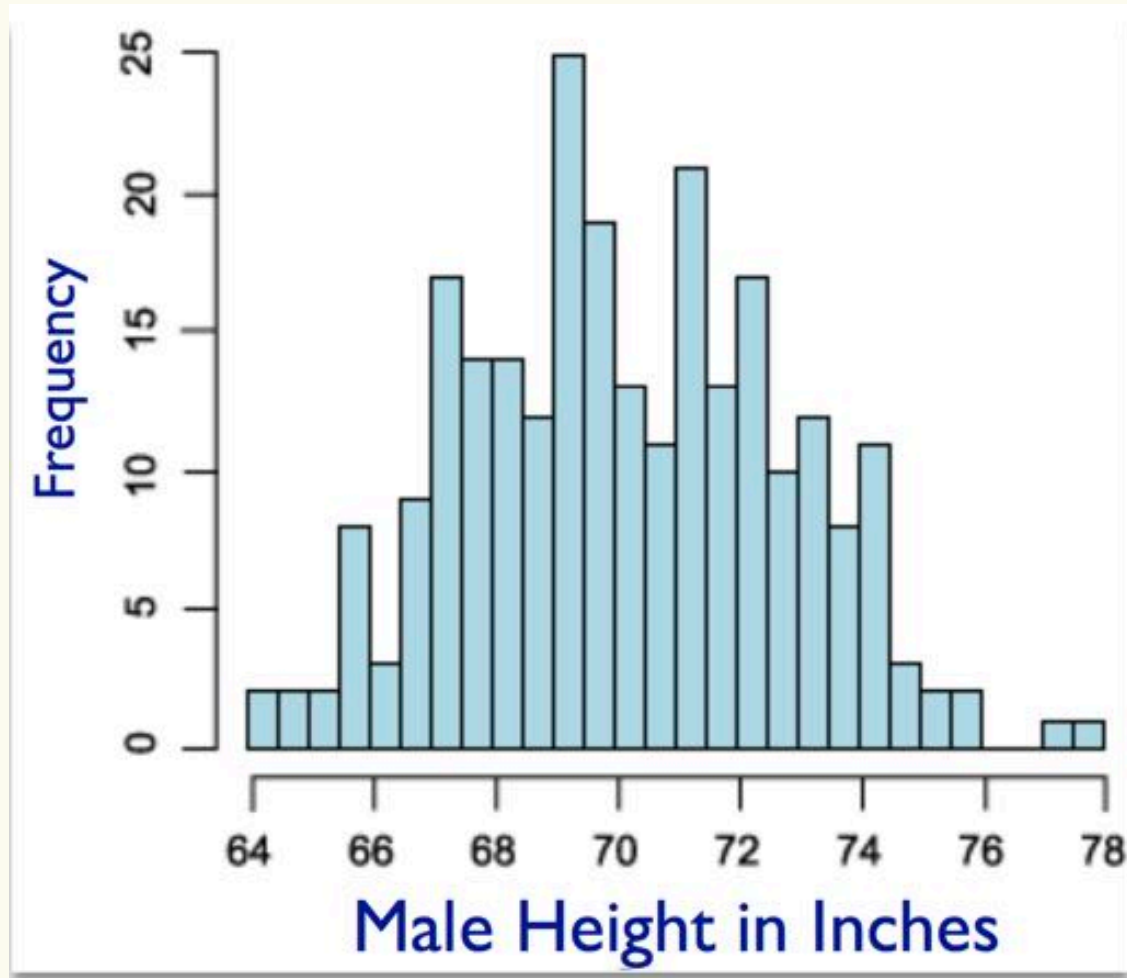
e.g. $k = 1$: 68%, $k = 2$: 95%, $k = 3$: 99%

Worse estimate with Chebyshev:

$$\mathbb{P}(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

Gaussian in Nature

Empirical distribution of collected data often resembles a Gaussian ...



e.g. Height distribution resembles Gaussian.

R.A.Fisher (1918) observed that the height is likely the outcome of the sum of many independent random parameters, i.e., can be written as

$$X = X_1 + \dots + X_n$$

Sum of Independent RVs

Reminder: i.i.d. = independent and identically distributed

X_1, \dots, X_n i.i.d. with expectation μ and variance σ^2

Define

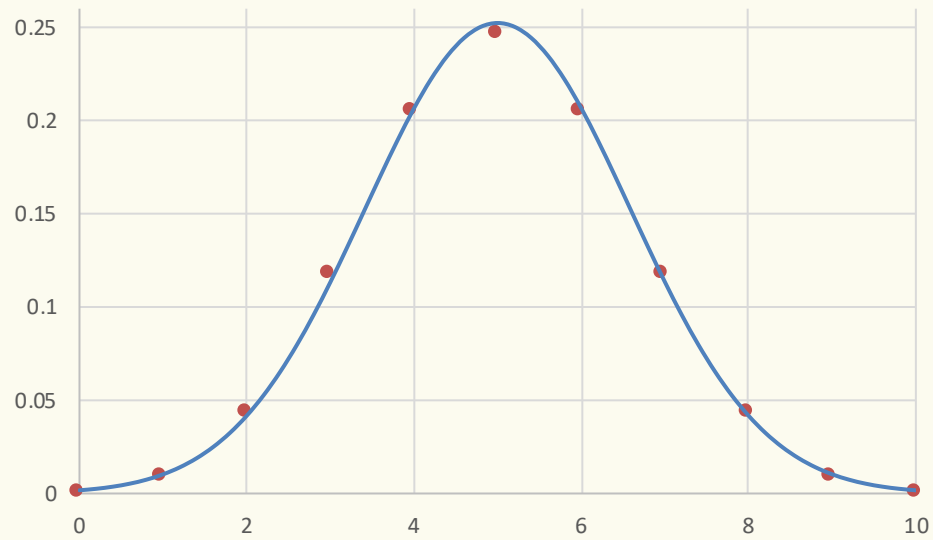
$$S_n = X_1 + \dots + X_n$$

$$\mathbb{E}(S_n) = \mathbb{E}(X_1) + \dots + \mathbb{E}(X_n) = n\mu$$

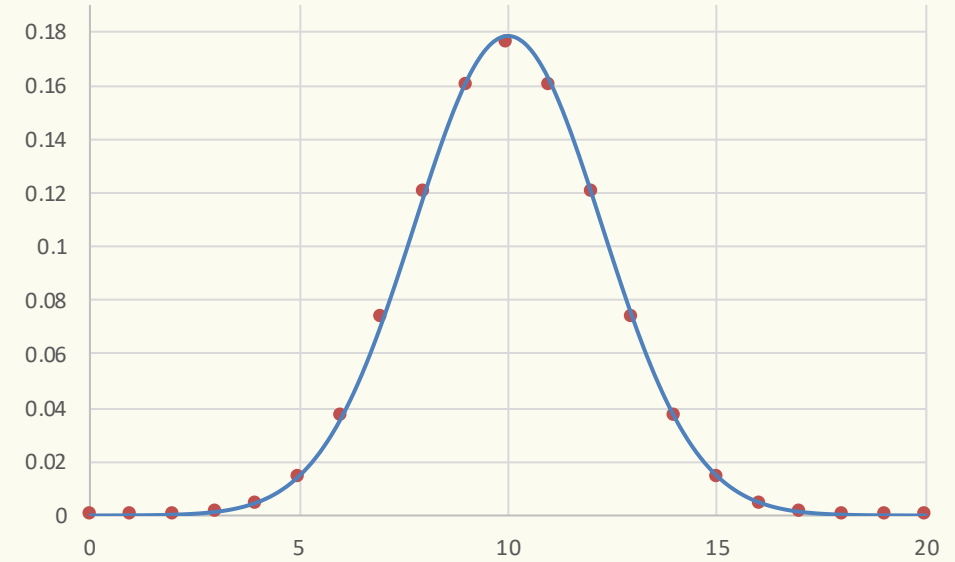
$$\text{Var}(S_n) = \text{Var}(X_1) + \dots + \text{Var}(X_n) = n\sigma^2$$

Empirical observation: S_n is very close to $\mathcal{N}(n\mu, n\sigma^2)$ as n grows.

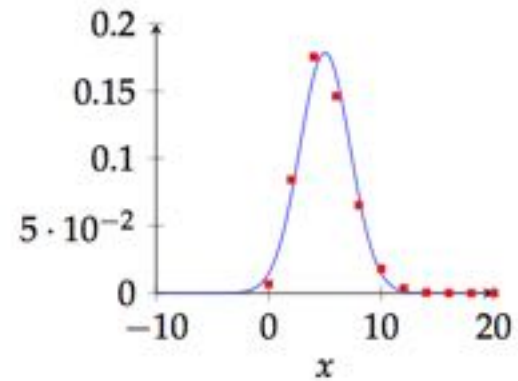
Example. $S_n = X_1 + \dots + X_n$, X_i Bernoulli with parameter $\frac{1}{2}$



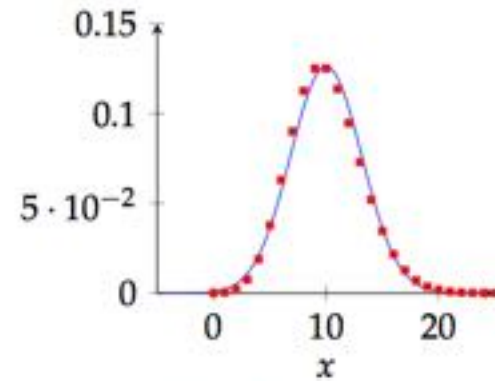
$$n = 10, n\mu = 5, n\sigma^2 = 2.5$$



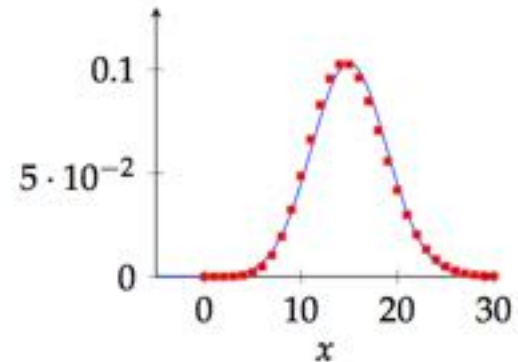
$$n = 20, n\mu = 10, n\sigma^2 = 5$$



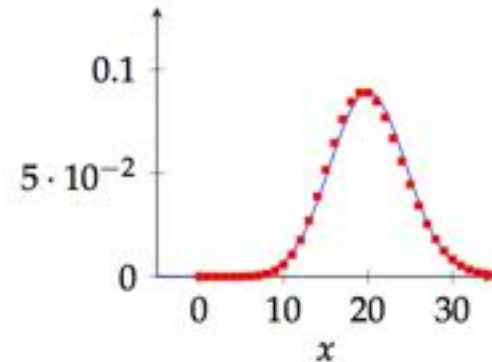
(a) $\lambda = 5$.



(b) $\lambda = 10$.

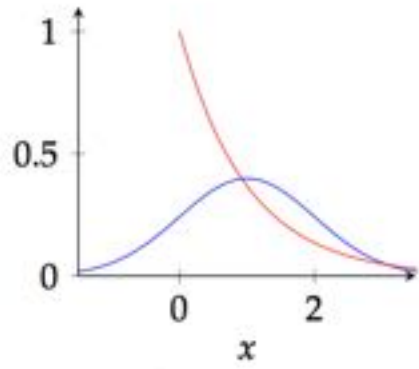


(c) $\lambda = 15$.

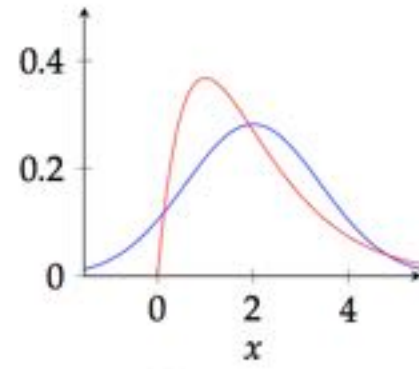


(d) $\lambda = 20$.

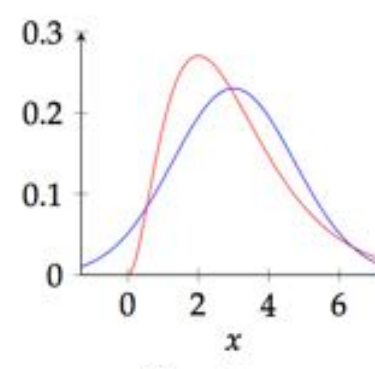
Sum of n Poisson RVs with parameter $\lambda = 1 =$ Poisson of with parameter $\lambda = n$ (See Quiz Sections)



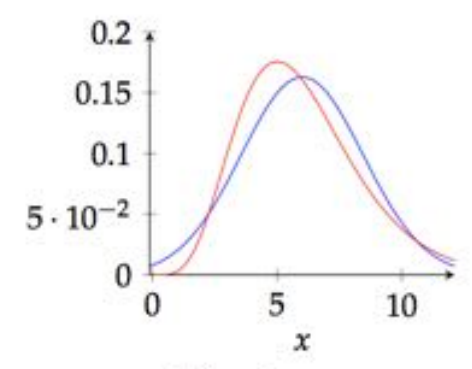
(a) $n = 1$



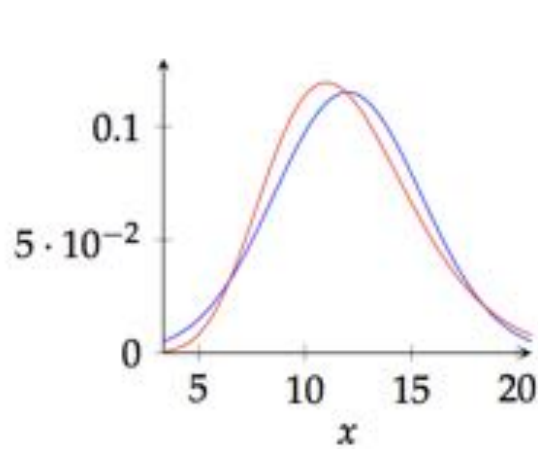
(b) $n = 2$



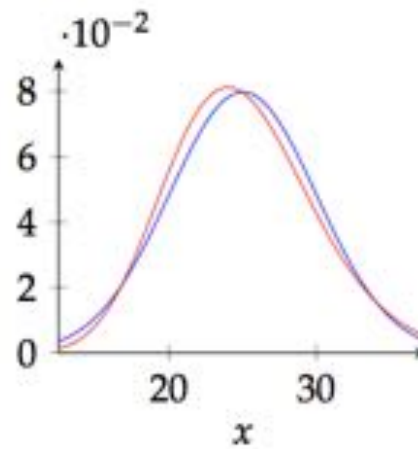
(c) $n = 3$



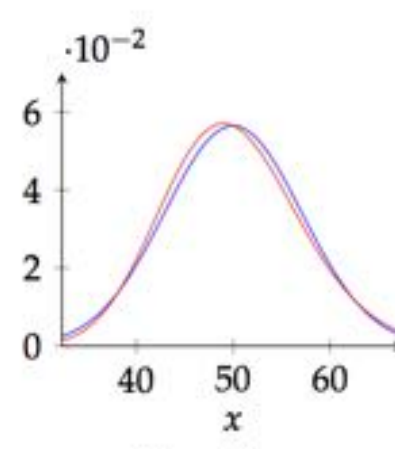
(d) $n = 6$



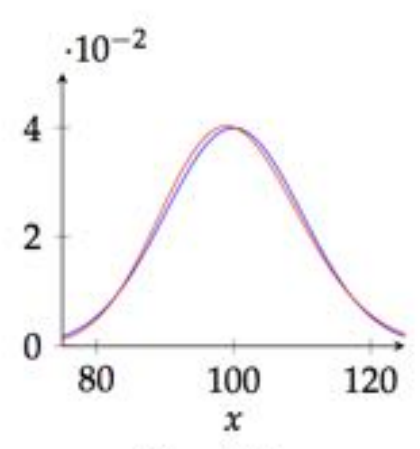
(e) $n = 12$



(f) $n = 25$



(g) $n = 50$



(h) $n = 100$

Sum of n exponentials with parameter $\lambda = 1$ (Erlang Distribution)

Central Limit Theorem

X_1, \dots, X_n i.i.d., each with expectation μ and variance σ^2

Define $S_n = X_1 + \dots + X_n$ and

$$Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

$$\mathbb{E}(Y_n) = \frac{1}{\sigma\sqrt{n}} (\mathbb{E}(S_n) - n\mu) = \frac{1}{\sigma\sqrt{n}} (n\mu - n\mu) = 0$$

$$\text{Var}(Y_n) = \frac{1}{\sigma^2 n} (\text{Var}(S_n - \mu n)) = \frac{\text{Var}(S_n)}{\sigma^2 n} = \frac{\sigma^2 n}{\sigma^2 n} = 1$$

Central Limit Theorem

$$Y_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

Theorem. (Central Limit Theorem) The CDF of Y_n converges to the CDF of the standard normal $\mathcal{N}(0,1)$, i.e.,

$$\lim_{n \rightarrow \infty} \mathbb{P}(Y_n \leq y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-x^2/2} dx$$

Example

We flip n independent coins, heads with probability $p = 0.75$.

$$X = \# \text{ heads} \quad \mu = \mathbb{E}(X) = 0.75n \quad \sigma^2 = \text{Var}(X) = 0.1875n$$

$$\mathbb{P}(X \leq 0.7n)$$

n	exact	$\mathcal{N}(\mu, \sigma^2)$ approx
10	0.4744072	0.357500327
20	0.38282735	0.302788308
50	0.25191886	0.207108089
100	0.14954105	0.124106539
200	0.06247223	0.051235217
1000	0.00019359	0.000130365