

CSE 312

Foundations of Computing II

Lecture 20: Continuous Random Variables



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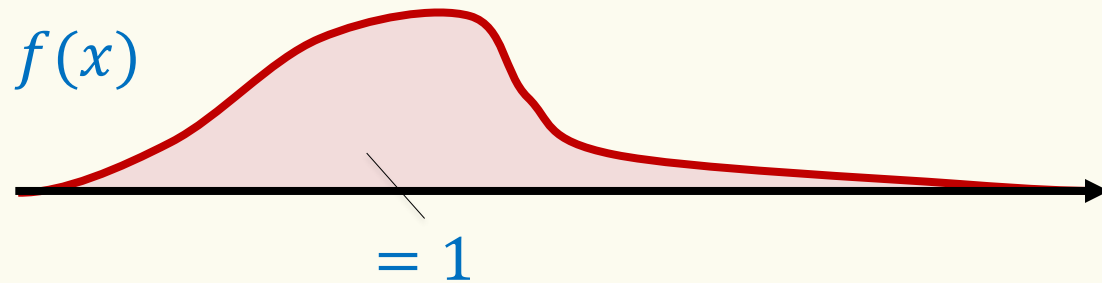
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Review – Continuous RVs

Probability Density Function (PDF).

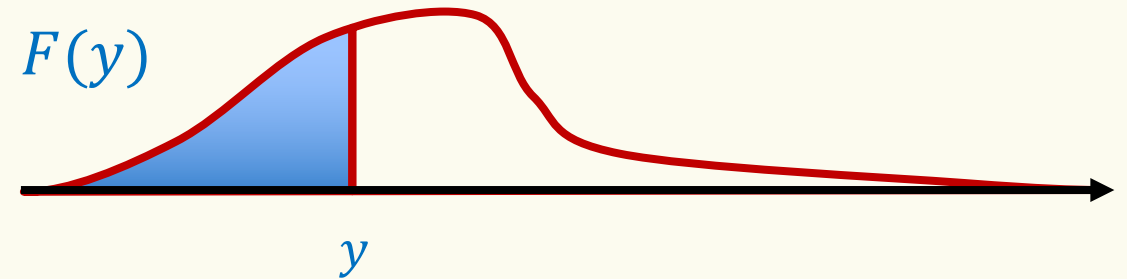
$f: \mathbb{R} \rightarrow \mathbb{R}$ s.t.

- $f(x) \geq 0$ for all $x \in \mathbb{R}$
- $\int_{-\infty}^{+\infty} f(x) dx = 1$



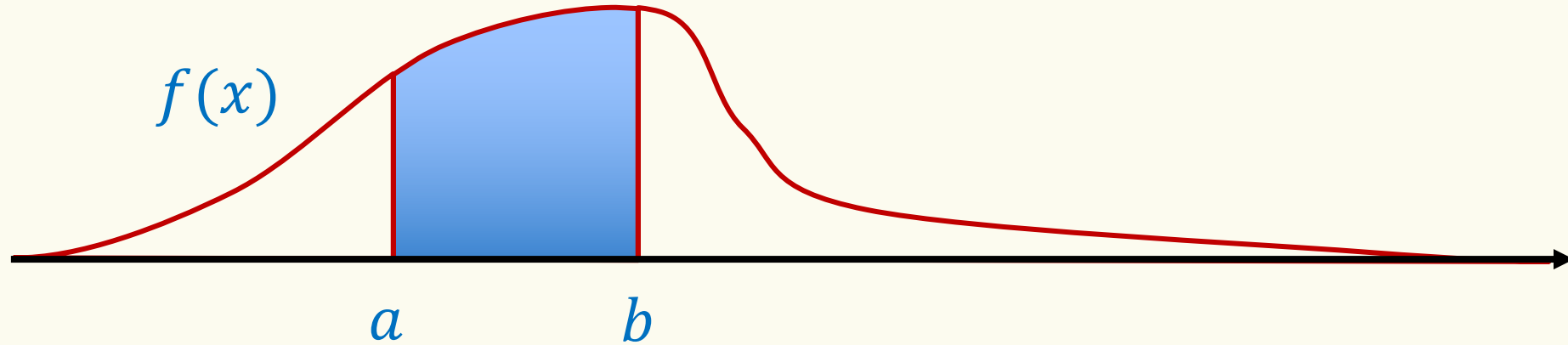
Cumulative Density Function (CDF).

$$F(y) = \int_{-\infty}^y f(x) dx$$



Theorem. $f(x) = \frac{dF(x)}{dx}$

Review – Continuous RVs



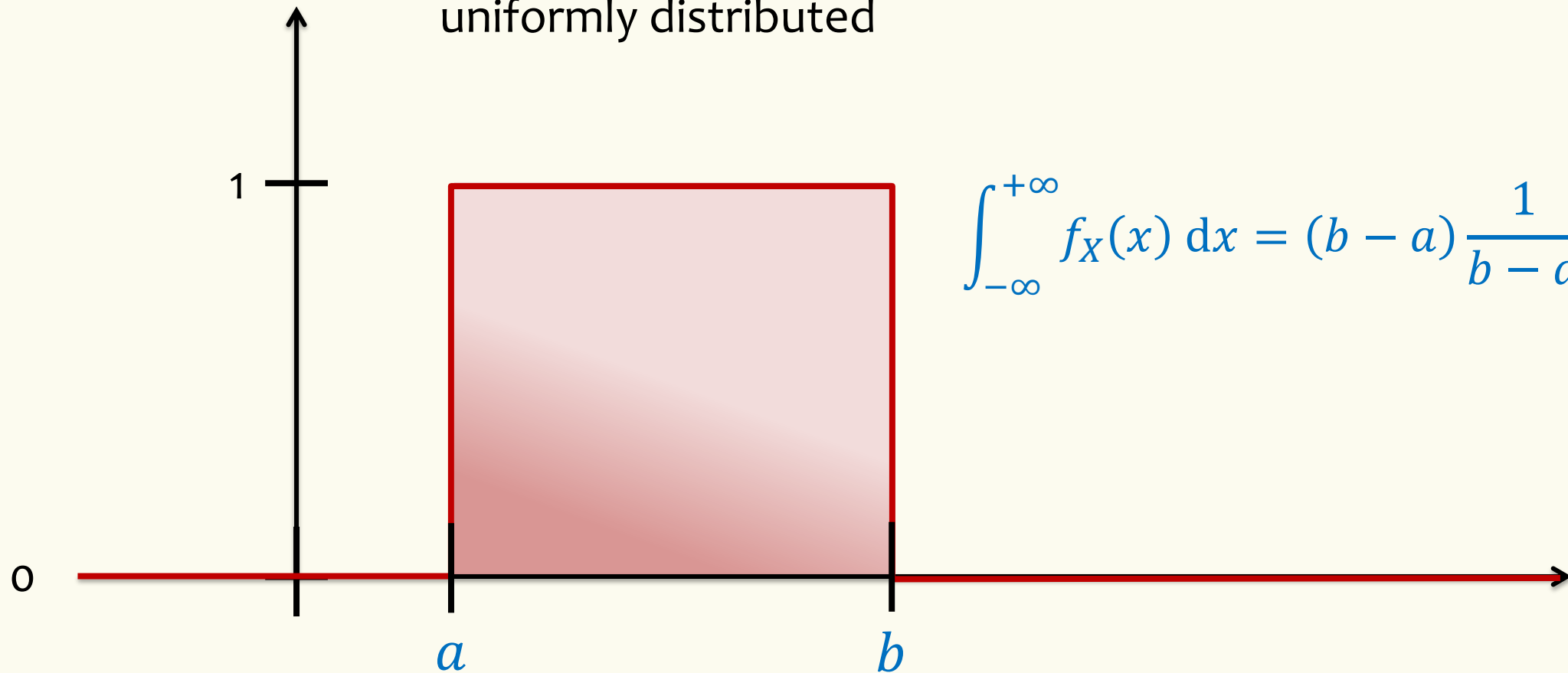
$$\mathbb{P}(X \in [a, b]) = \int_a^b f(x) dx = F(b) - F(a)$$

Uniform Distribution

$$X \sim \text{Unif}(a, b)$$

We also say that X follows the uniform distribution / is uniformly distributed

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$



$$\int_{-\infty}^{+\infty} f_X(x) dx = (b-a) \frac{1}{b-a} = 1$$

Uniform Density – Expectation

$$X \sim \text{Unif}(a, b)$$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, dx$$

$$\begin{aligned} &= \frac{1}{b-a} \int_a^b x \, dx = \frac{1}{b-a} \left(\frac{x^2}{2} \right) \Big|_a^b \\ &= \frac{1}{b-a} \left(\frac{b^2 - a^2}{2} \right) = \frac{(b-a)(a+b)}{2(b-a)} = \frac{a+b}{2} \end{aligned}$$

Uniform Density – Variance

$$X \sim \text{Unif}(a, b)$$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{+\infty} f_X(x) \cdot x^2 \, dx$$

$$= \frac{1}{b-a} \int_a^b x^2 \, dx = \frac{1}{b-a} \left(\frac{x^3}{3} \right) \Big|_a^b$$

$$= \frac{b^3 - a^3}{3(b-a)} = \frac{(b-a)(b^2 + ab + a^2)}{3(b-a)} = \frac{b^2 + ab + a^2}{3}$$

Uniform Density – Variance

$$\mathbb{E}(X^2) = \frac{b^2 + ab + a^2}{3} \quad \mathbb{E}(X) = \frac{a + b}{2}$$

$$X \sim \text{Unif}(a, b)$$

$$\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$= \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4}$$

$$= \frac{4b^2 + 4ab + 4a^2}{12} - \frac{3a^2 + 6ab + 3b^2}{12}$$

$$= \frac{b^2 - 2ab + a^2}{12} = \frac{(b - a)^2}{12}$$

Exponential Density

Assume expected # of occurrences of an event per unit of time is λ

- Cars going through intersection
- Number of lightning strikes
- Requests to web server
- Patients admitted to ER

Numbers of occurrences of event: Poisson distribution

$$\mathbb{P}(X = i) = e^{-\lambda} \frac{\lambda^i}{i!} \quad (\text{Discrete})$$

How long to wait until next event? Exponential density!

Let's define it and then derive it!

Exponential Distribution

Definition. An **exponential random variable** X with parameter $\lambda \geq 0$ is follows the exponential density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

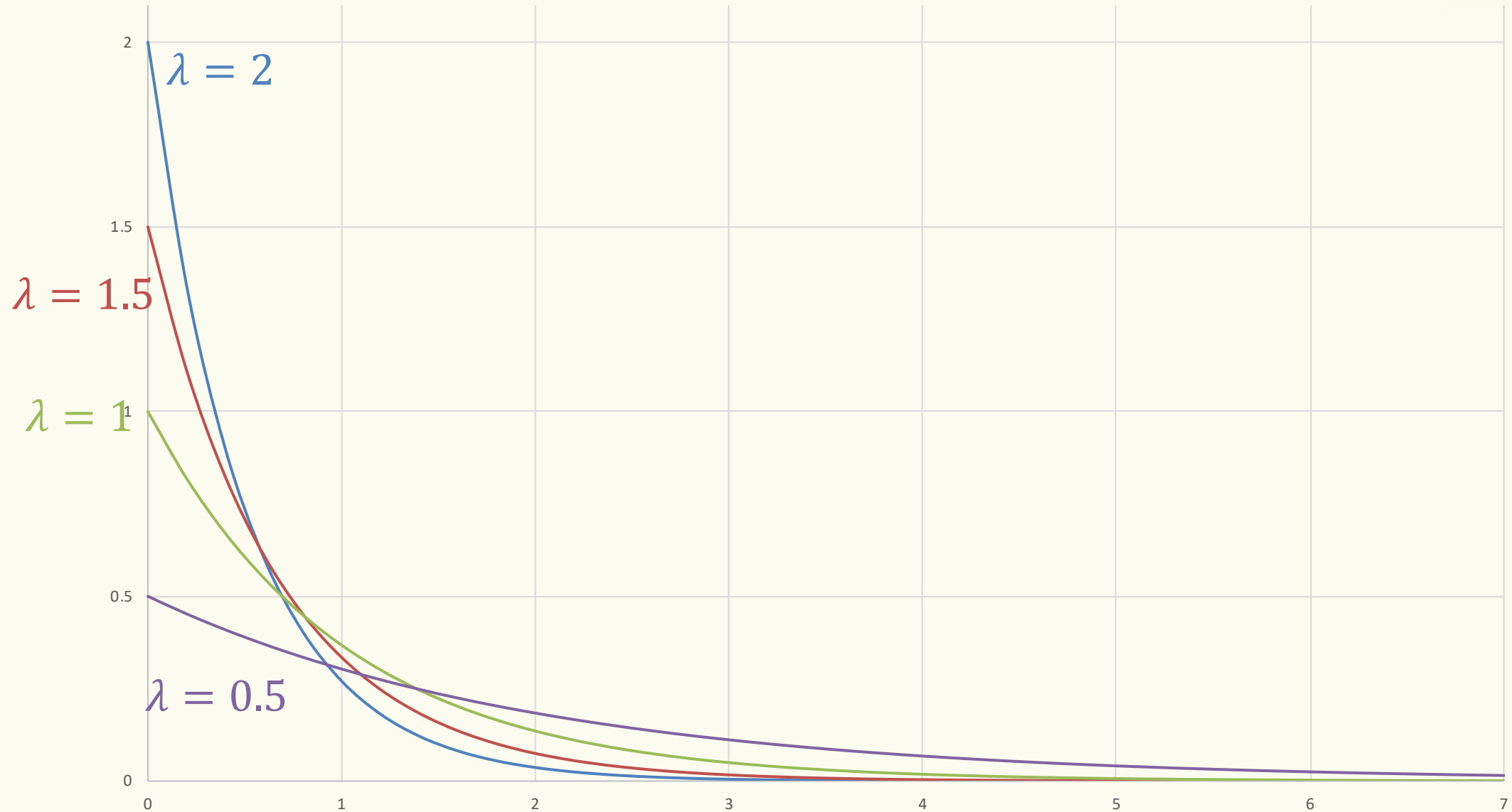
We write $X \sim \text{Exp}(\lambda)$ and say X that follows the exponential distribution.

CDF: For $y \geq 0$,

$$F_X(y) = \int_0^y \lambda e^{-\lambda x} dx = \lambda \left((-1/\lambda) e^{-\lambda x} \right) \Big|_0^y = 1 - e^{-\lambda y}$$

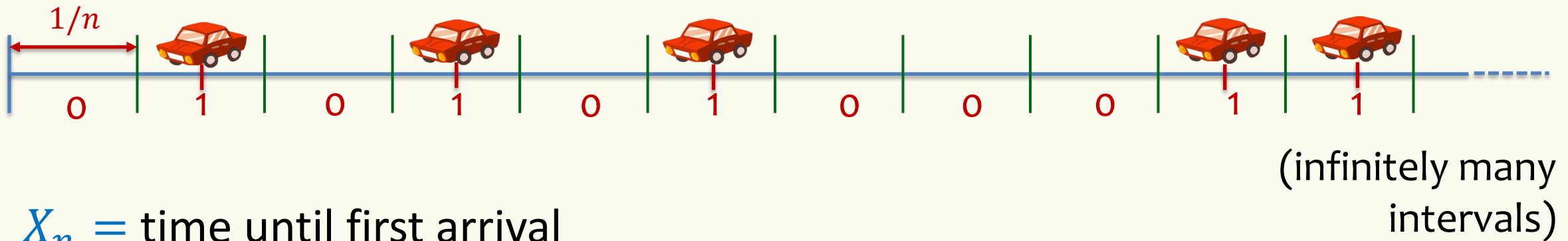
Exponential Distribution

[Densities are all 0 on negative reals]



Derivation – Number of Cars

Discretize: In each interval of length $1/n$, probability $p = \frac{\lambda}{n}$ of car arriving.



$$\mathbb{P}(X_n > y) = \left(1 - \frac{\lambda}{n}\right)^{yn} \quad \text{(scaled) geometric}$$

$$\lim_{n \rightarrow \infty} \mathbb{P}(X_n > y) = \lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^{yn} = e^{-\lambda y} = 1 - F_X(y)$$

Expectation

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\begin{aligned} \mathbb{E}(X) &= \int_{-\infty}^{+\infty} f_X(x) \cdot x \, dx \\ &= \int_0^{+\infty} \lambda x e^{-\lambda x} \cdot x \, dx \\ &= \lim_{r \rightarrow \infty} \left(-\left(x + \frac{1}{\lambda}\right) e^{-\lambda x} \right) \Big|_0^r = \frac{1}{\lambda} \end{aligned}$$

$$\mathbb{E}(X) = \frac{1}{\lambda}$$

$$\text{Var}(X) = \frac{1}{\lambda^2}$$

Memorylessness

Definition. A random variable is **memoryless** if for all $s, t > 0$,

$$\mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t).$$

Fact. $X \sim \text{Exp}(\lambda)$ is memoryless.

Assuming exp distr, if you've waited s minutes,
prob of waiting t more is exactly same as $s = 0$

Assuming exp distr, if you've waited s minutes, prob of waiting t more is exactly same as $s = 0$

Memorylessness of Exponential

Fact. $X \sim \text{Exp}(\lambda)$ is memoryless.

Proof.

$$\begin{aligned}\mathbb{P}(X > s + t \mid X > s) &= \frac{\mathbb{P}(\{X > s + t\} \cap \{X > s\})}{\mathbb{P}(X > s)} \\ &= \frac{\mathbb{P}(X > s + t)}{\mathbb{P}(X > s)} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = \mathbb{P}(X > t)\end{aligned}$$

The Normal Distribution

Definition. A **Gaussian (or normal) random variable** with parameters $\mu \in \mathbb{R}$ and $\sigma \geq 0$ has density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

(We say that X follows the Normal Distribution, and write $X \sim \mathcal{N}(\mu, \sigma^2)$)

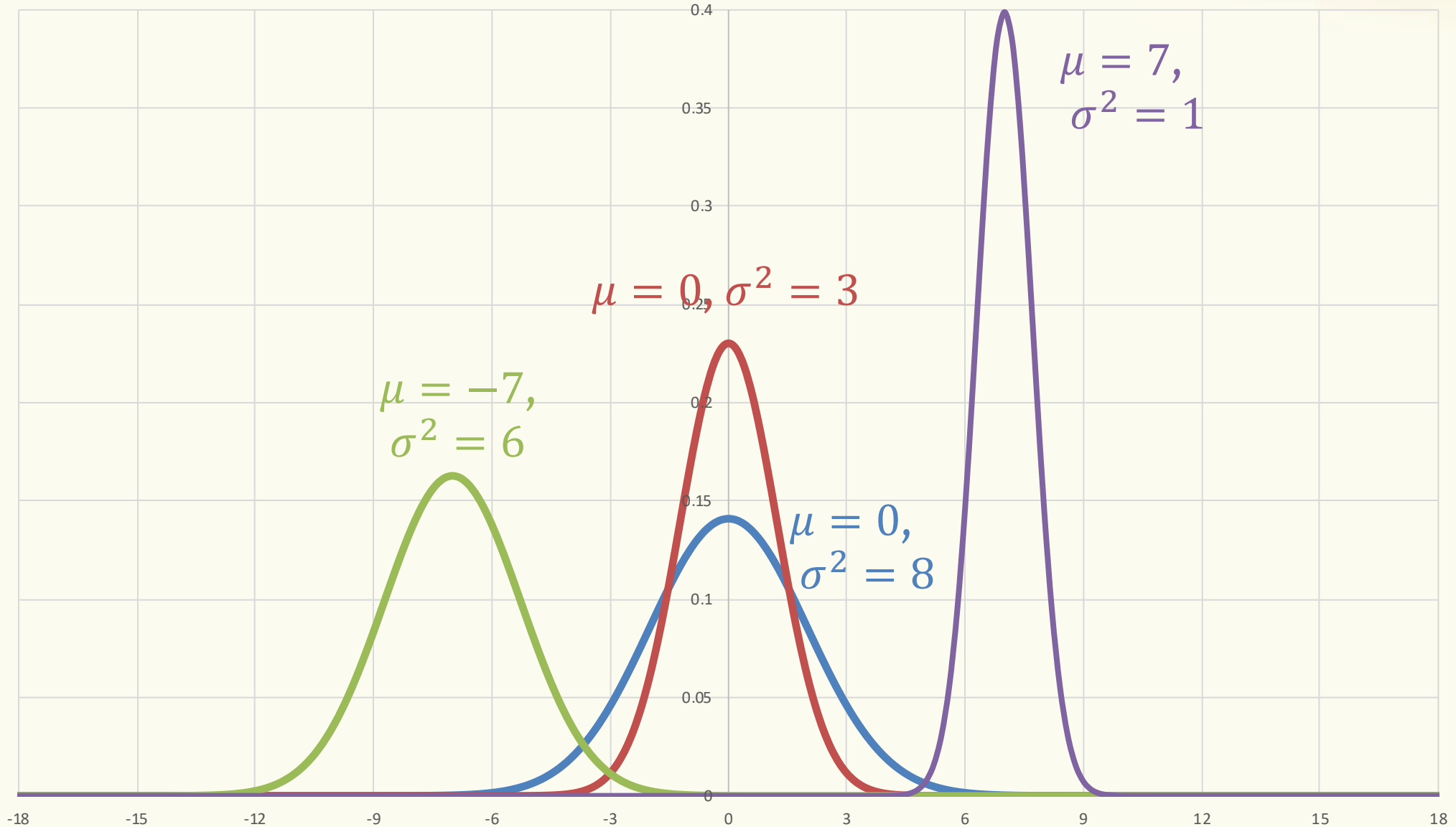
We will see next time why the normal distribution is (in some sense) the most important distribution.



Carl Friedrich
Gauss

The Normal Distribution

Aka a “Bell Curve” (imprecise name)



Two Facts

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\mathbb{E}(X) = \mu$

Proof is easy because density curve is symmetric, and $x \cdot f_X(\mu - x) = x \cdot f_X(\mu + x)$

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\text{Var}(X) = \sigma^2$

Shifting and Scaling

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

How do we prove this? (Likely) see next week for a simple proof.

Standard (unit) normal = $\mathcal{N}(0, 1)$

Definition. $\Phi(z) = \mathbb{P}(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-x^2/2} dx$ for $Z \sim \mathcal{N}(0, 1)$

Note: $\Phi(z)$ has no closed form – generally given via tables

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F_X(z) = \mathbb{P}(X \leq z) = \mathbb{P}\left(\frac{X-\mu}{\sigma} \leq \frac{z-\mu}{\sigma}\right) = \Phi\left(\frac{z-\mu}{\sigma}\right)$

Table of Standard Cumulative Normal Density

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451