CSE 312

Foundations of Computing II

Lecture 2: Permutations and Combinations

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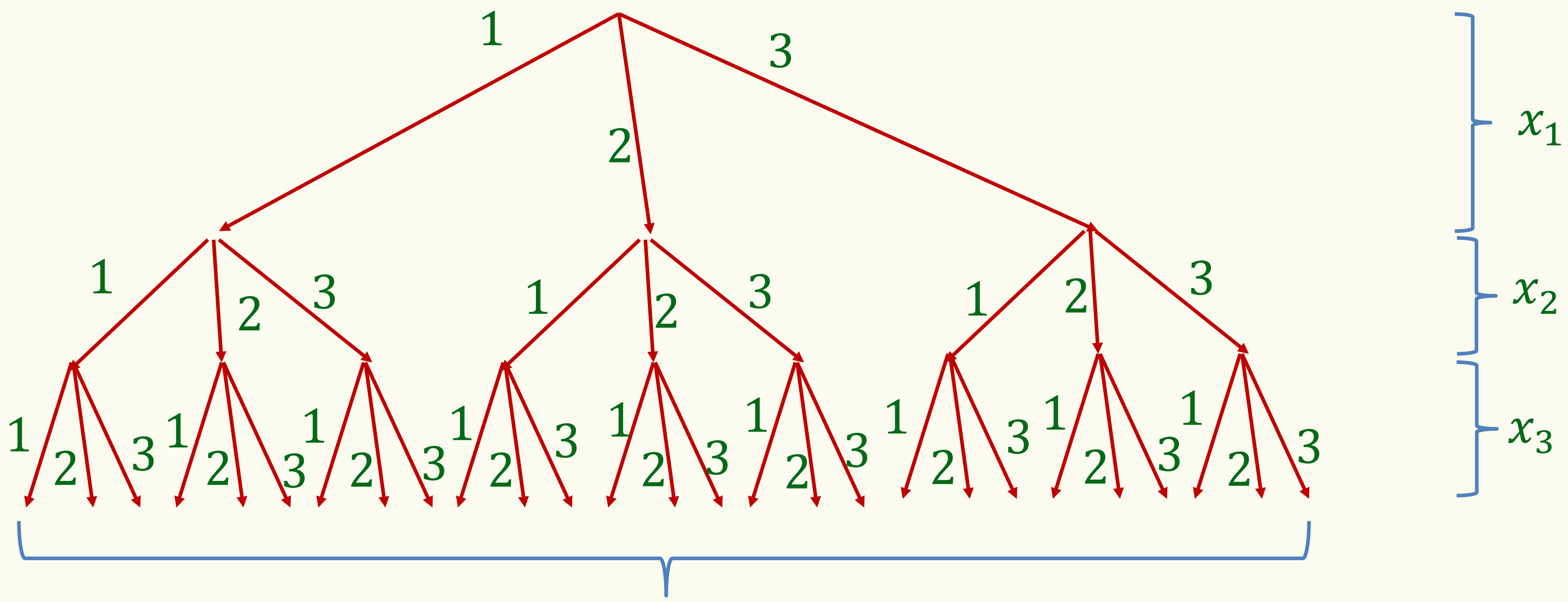
Sequential process: We fix elements in a sequence one by one, and see how many possibilities we have at each step.

Example: "How many sequences are there in $\{1,2,3\}^3$?"

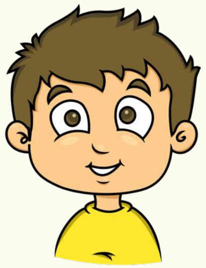


27 paths = 27 sequences

Product rule – One more example

5 books

Alice



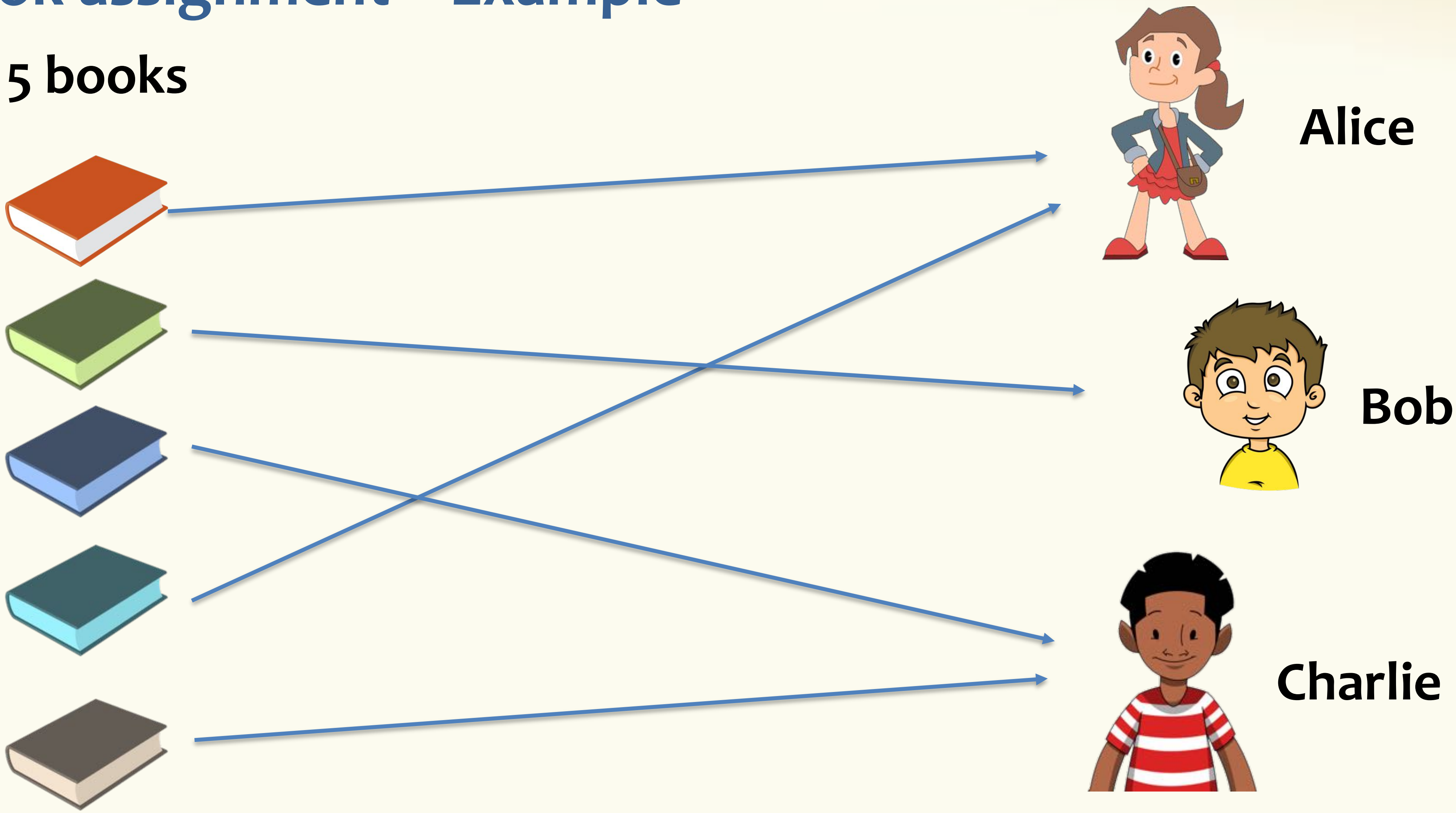
Bob

Charlie

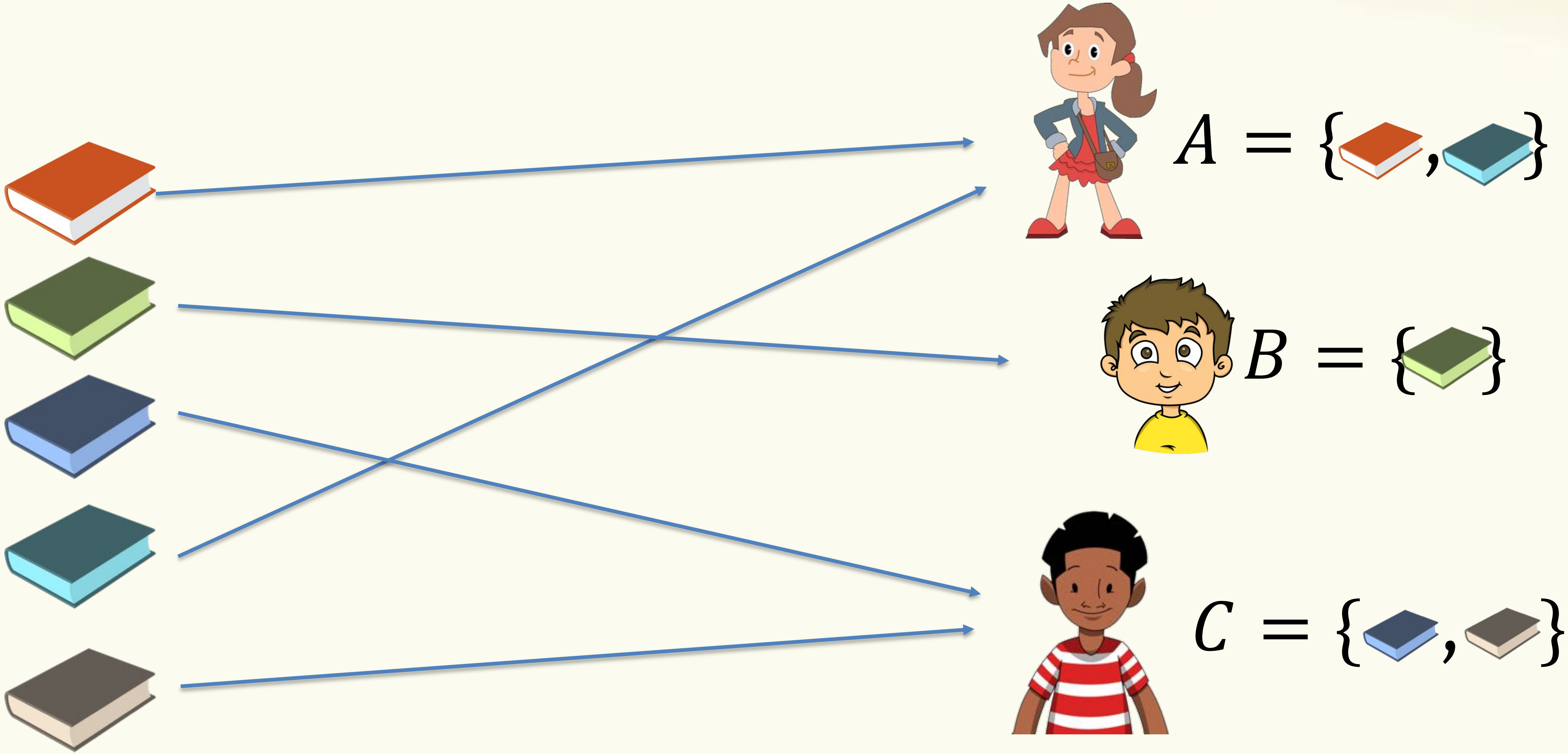
"How many ways are there to distribute 5 books among Alice, Bob, and Charlie?"

Every book to one person, everyone gets ≥ 0 books.

Book assignment – Example



Book assignment – Counting

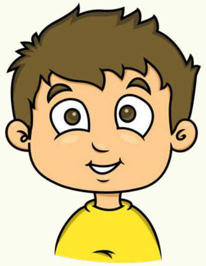


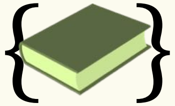
Book assignment – Modeling

$2^5 = 32$ options

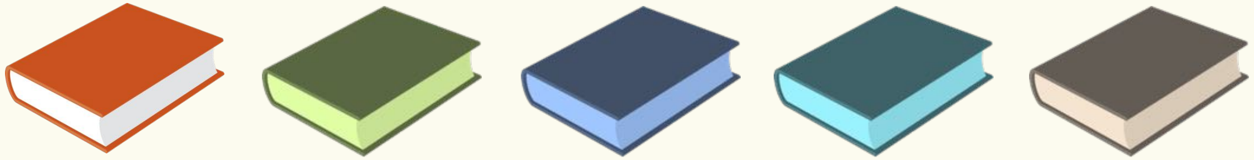
Assignment = (A, B, C)

$A = \{$  $\}$



$B = \{$  $\}$

$C = \{$  $\}$



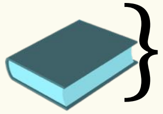
Book assignments – Modeling

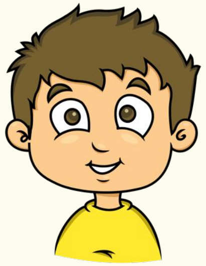
$2^5 = 32$ options

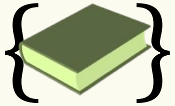
Assignment = (A, B, C)

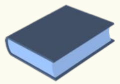
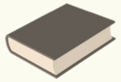
assignments =
$2^5 \times 2^5 \times 2^5 = 2^{15} = 32768$

Correct?

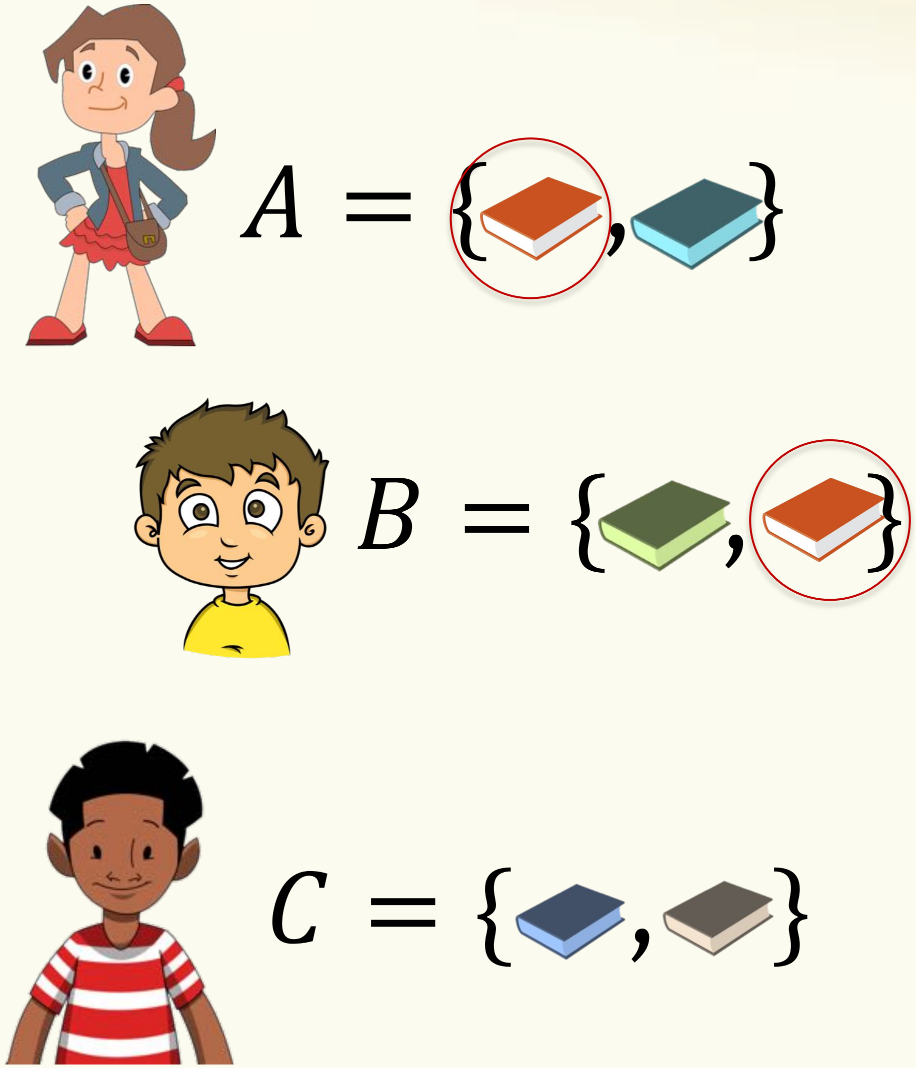
$A = \{$  $,$  $\}$



$B = \{$  $\}$

$C = \{$  $,$  $\}$

Problem – Overcounting

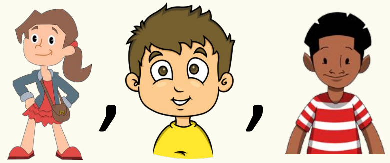


Assignment = (A, B, C)

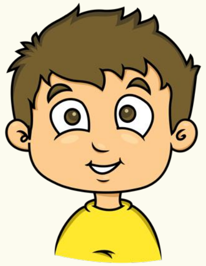
Problem: We are counting some <u>invalid</u> assignments!!!
$\longrightarrow$ <u>**overcounting**</u>!

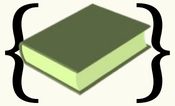
Book assignments – Correct Approach

Assignment = $(x_1, \ldots, x_5)$

$x_i \in \{$  $\}$

$A = \{$  $\}$

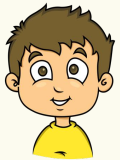


$B = \{$  $\}$

$C = \{$  $\}$

Book assignments – Correct Approach

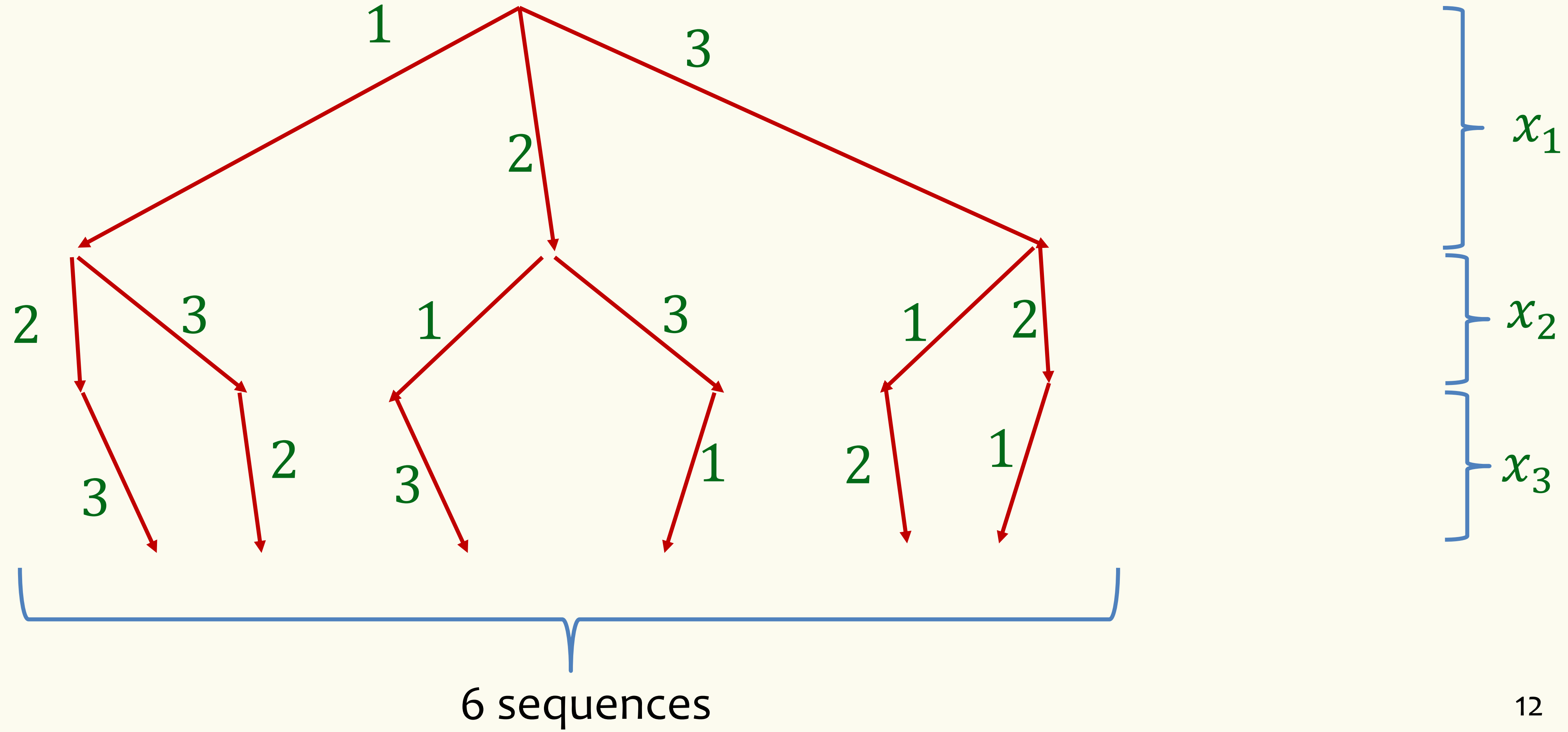
Assignment = $(x_1, \ldots, x_5)$

$x_i \in \{$ , ,  $\}$

assignments = $3^5 = 243$

Representation of what we are counting is very important!

"How many sequences in $\{1,2,3\}^3$ with no repeating elements?"



Factorial

"How many ways to order elements in S, where $|S| = n$?"

Permutations

Answer = $n\times(n-1)\times(n-2)\times\cdots\times2\times1$

Definition. The **factorial function** is

$$n! = n\times(n-1)\times\cdots\times2\times1$$

Note: $0! = 1$

Theorem. (Stirling's approximation)

$$\underbrace{\sqrt{2\pi}}_{=\,2.5066}\cdot n^{n+\frac{1}{2}}\cdot e^{-n} \le n! \le \underbrace{e}_{=\,2.7183}\cdot n^{n+\frac{1}{2}}\cdot e^{-n}$$

Distinct Letters

"How many sequences of 5 *distinct alphabet letters from* $\{A, B, \ldots, Z\}$*?"*

E.g., AZURE, BINGO, TANGO. But not: STEVE, SARAH

Answer: $26 \times 25 \times 24 \times 23 \times 22 = 7893600$

In general

Aka: k-permutations

Fact. # of k-element sequences of distinct symbols from n-element set is

$$P(n,k) = n\times(n-1)\times\cdots\times(n-k+1) = \frac{n!}{(n-k)!}$$

Number of Subsets

"How many size-5 ***subsets*** *of* $\{A, B, \ldots, Z\}$*?"*

E.g., {A,Z,U,R,E}, {B,I,N,G,O}, {T,A,N,G,O}. But not: {S,T,E,V}, {S,A,R,H},…

Number of Subsets – Idea

To generate all sequences of 5 distinct letters from $\{A, B, \dots, Z\}$: $\rightarrow \dfrac{26!}{21!}$

- Go through all sets $S \subseteq \{A, B, \dots, Z\}$ of size $|S| = 5$ $\rightarrow$???
  e.g. $S = \{A, G, N, O, T\}$
  - Go through all permutations of S $\rightarrow$ $5!$
    e.g., *TANGO, AGNOT, NAGOT, GOTAN, GOATN, NGOAT, …*

Product rule: $\dfrac{26!}{21!} = ??? \times 5! \rightarrow ??? = \dfrac{26!}{21!\,5!} = 65780$

Number of Subsets – Binomial Coefficient

Fact. The number of subsets of size k of a set of size n is

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Binomial coefficient (verbalized as "n choose k")

Notation: $\binom{S}{k}$ = all k-element subsets of S $\qquad \left|\binom{S}{k}\right| = \binom{|S|}{k}$

[also called **combinations**]

Binomial Coefficient – Properties

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\binom{n}{n} = 1 \qquad \binom{n}{1} = n \qquad \binom{n}{0} = 1$$

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ **Pascal's Identity**

Fact. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

Example I – Sum of integers

"How many solutions $(x_1, \dots, x_k)$ *such that* $x_1, \dots, x_k \geq 0$ *and* $\sum_{i=1}^{k} x_i = n$*?"*

Example: $k = 3$, $n = 5$

$(0,0,5), (5,0,0), (1,0,4), (2,1,2), (3,1,1), (2,3,0), \dots$

Example I – Sum of integers

Example: $k = 3$, $n = 5$

$$(0,0,5), (5,0,0), (1,0,4), (2,1,2), (3,1,1), (2,3,0), \ldots$$

Clever encoding of solutions

$$(3,1,1) \quad (2,1,2) \quad (1,0,4)$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$1\,1\,1\,0\,1\,0\,1 \qquad 1\,1\,0\,1\,0\,1\,1 \qquad 1\,0\,0\,1\,1\,1\,1$$

Example I – Sum of integers

Example: $k = 3, n = 5$

sols = # els. from $\{0,1\}^7$ w/ exactly two 0s $= \binom{7}{2} = 21$

Clever encoding of solutions

$$(3,1,1) \downarrow 1\,1\,1\,0\,1\,0\,1$$

$$(2,1,2) \downarrow 1\,1\,0\,1\,0\,1\,1$$

$$(1,0,4) \downarrow 1\,0\,0\,1\,1\,1\,1$$

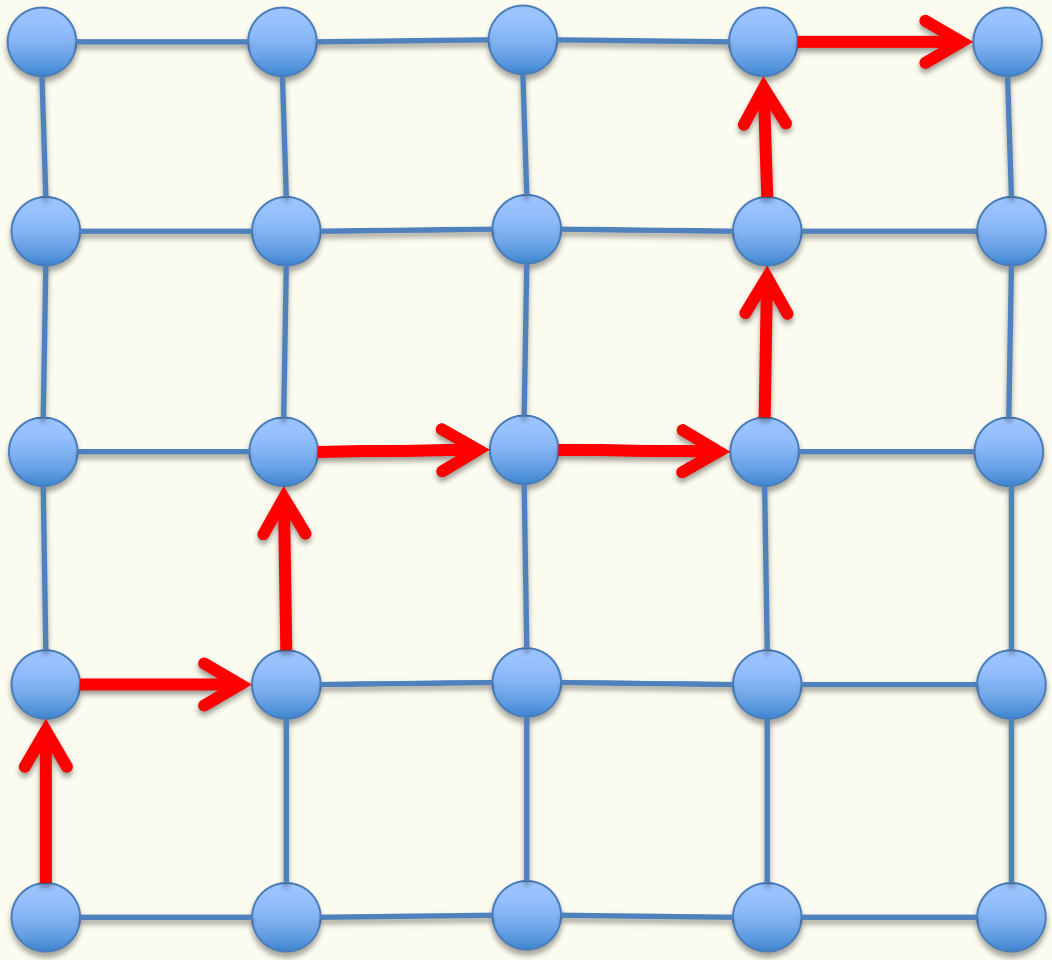
Example I – Sum of integers

"How many solutions $(x_1, \dots, x_k)$ *such that* $x_1, \dots, x_k \geq 0$ *and* $\sum_{i=1}^{k} x_i = n$*?"*

sols = # els. from $\{0,1\}^{n+k-1}$ w/ $k-1$ 0s

$$= \binom{n+k-1}{k-1}$$

Example II – Counting Paths



"How many shortest paths from bottom-left to top-right?"