CSE 312

Foundations of Computing II

Lecture 13: Variance and Independent Random Variables

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Functions of Random Variables

Sometimes, a random variable is a function of one or more other random variables. Examples:

- $X + Y$
- X^2
- $X^2 + XY$
- $\sqrt{X}$
- $e^X Y^2$

$$\boxed{Z = g(X, Y, \ldots)}$$

Bottom line: You can compute with them, but handle with care!

Functions of Random Variables – Example 1

Let X be the outcome of a fair die.

- Then, X^2 is the random variable which takes values $1,4,9,16,25,36$ with probability $\frac{1}{6}$ each.

$$\mathbb{E}(X^2) = \sum_x x^2 \cdot \mathbb{P}(X = x)$$

$$= \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) = \frac{91}{6}$$

Functions of Random Variables – Example 2

Let X, Y be outcomes of two dice rolls

$$\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y) \cdot \mathbb{P}(X = x, Y = y)$$

$$\mathbb{E}[(X+2Y)^2] = \textstyle\sum_{x,y}(x+2y)^2 \cdot \mathbb{P}(X = x, Y = y) = \cdots$$

Also note: $(X+2Y)^2 = X^2 + 4XY + 4Y^2$, and thus

$$\mathbb{E}[(X+2Y)^2] = \mathbb{E}[X^2 + 4XY + 4Y^2] = \mathbb{E}(X^2) + 4\mathbb{E}(XY) + 4\mathbb{E}(Y^2)$$

(by linearity)

Two Games

Game 1: *In every round, you win \$2 with probability 1/3, lose \$1 with probability 2/3.*

W_1 = payoff in a round of Game 1

$$\mathbb{P}(W_1 = 2) = \frac{1}{3}, \mathbb{P}(W_1 = -1) = \frac{2}{3}$$

$$\mathbb{E}(W_1) = 0$$

Game 2: *In every round, you win \$10 with probability 1/3, lose \$5 with probability 2/3.*

W_2 = payoff in a round of Game 2

$$\mathbb{P}(W_2 = 10) = \frac{1}{3}, \mathbb{P}(W_2 = -5) = \frac{2}{3}$$

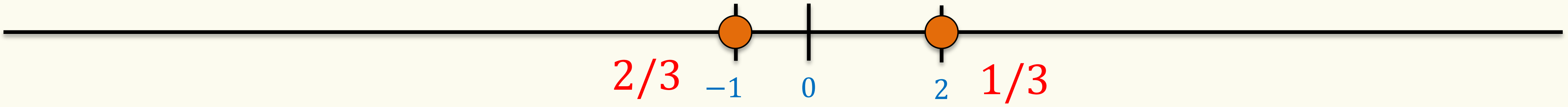
$$\mathbb{E}(W_2) = 0$$

Which game would you <u>rather</u> play?

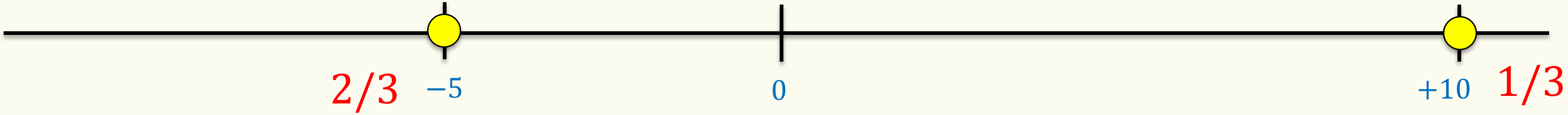
Somehow, Game 2 has higher volatility / exposure!

Two Games

$$\mathbb{P}(W_1 = 2) = \frac{1}{3}, \mathbb{P}(W_1 = -1) = \frac{2}{3}$$



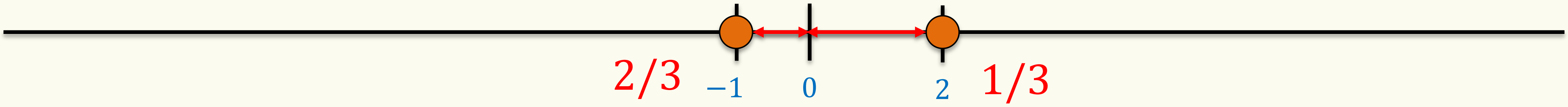
$$\mathbb{P}(W_2 = 10) = \frac{1}{3}, \mathbb{P}(W_2 = -5) = \frac{2}{3}$$



Two Games

$$\mathbb{E}(W_1) = 0$$

$$\mathbb{P}(W_1 = 2) = \frac{1}{3}, \mathbb{P}(W_1 = -1) = \frac{2}{3}$$



New quantity: How far off the expectation is the outcome?

$$\Delta(W_1) = |W_1 - \mathbb{E}(W_1)|$$

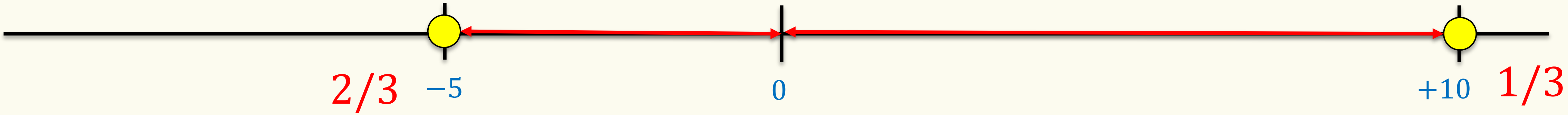
$$\mathbb{P}(\Delta(W_1) = 1) = \frac{2}{3}$$

$$\mathbb{P}(\Delta(W_1) = 2) = \frac{1}{3}$$

$$\mathbb{E}(\Delta(W_1)) = \frac{2}{3} \cdot 1 + \frac{1}{3} \cdot 2 = \frac{4}{3}$$

Two Games

$$\mathbb{P}(W_2 = 10) = \frac{1}{3}, \mathbb{P}(W_2 = -5) = \frac{2}{3}$$



New quantity: How far off the expectation is the outcome?

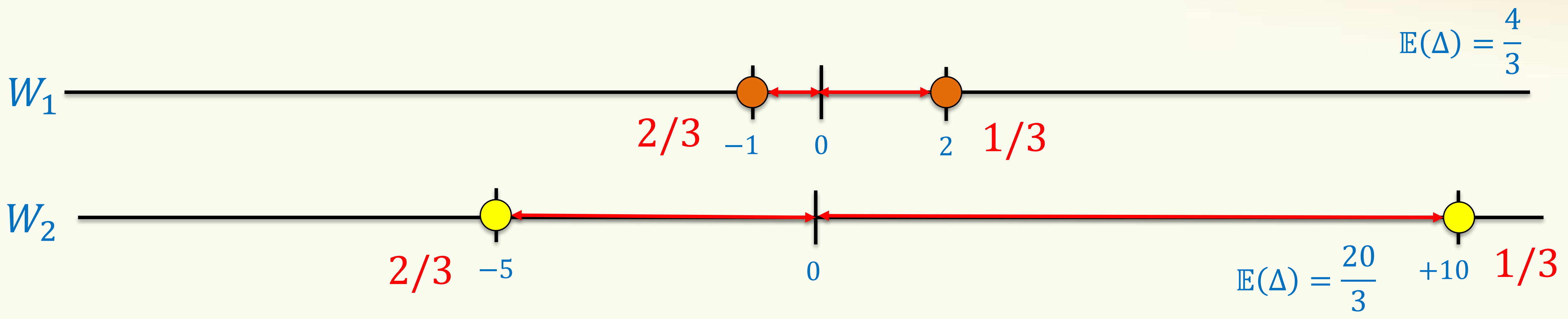
$$\Delta(W_2) = |W_2 - \mathbb{E}(W_2)|$$

$$\mathbb{P}(\Delta(W_2) = 5) = \frac{2}{3}$$

$$\mathbb{P}(\Delta(W_2) = 10) = \frac{1}{3}$$

$$\mathbb{E}(\Delta(W_2)) = \frac{2}{3} \cdot 5 + \frac{1}{3} \cdot 10 = \frac{20}{3}$$

Variance



We say that W_2 has **"higher variance"** than W_1.

However: Better to express deviation of outcome as $\left(X - \mathbb{E}(X)\right)^2$, instead of $|X - \mathbb{E}(X)|$ …

"Penalizes" big deviations + better analytical properties.

Variance

Definition. The **variance** of a (discrete) RV X is

$$\mathrm{Var}(X) = \mathbb{E}\left[\left(X - \mathbb{E}(X)\right)^2\right] = \sum_x \mathbb{p}_X(x) \cdot \left(x - \mathbb{E}(X)\right)^2$$

Standard deviation: $\sigma(X) = \sqrt{\mathrm{Var}(X)}$

Variance – Example 1

X fair die

- $\mathbb{P}(X = 1) = \cdots = \mathbb{P}(X = 6) = 1/6$
- $\mathbb{E}(X) = 3.5$

$$\mathrm{Var(X)} = \frac{1}{6}[(1-3.5)^2 + (2-3.5)^2 + (3-3.5)^2 + (4-3.5)^2 + (5-3.5)^2 + (6-3.5)^2]$$

$$= \frac{2}{6}[2.5^2 + 1.5^2 + 0.5^2] = \frac{2}{6}\left[\frac{25}{4} + \frac{9}{4} + \frac{1}{4}\right] = \frac{35}{12} \approx 2.91677 \ldots$$

Sections: General case + multiple rolls!

Variance – Example 2

X Bernoulli with parameter p

- $\mathbb{P}(X = 1) = p$, $\mathbb{P}(X = 0) = 1 - p$
- $\mathbb{E}(X) = p$

$$
\begin{aligned}
\mathrm{Var(X)} &= p\big(1 - \mathbb{E}(X)\big)^2 + (1-p)\big(0 - \mathbb{E}(X)\big)^2 \\
&= p(1-p)^2 + (1-p)p^2 \\
&= p(1-p)\big(p + (1-p)\big) = p(1-p)
\end{aligned}
$$

Variance – Properties

Definition. The **variance** of a (discrete) RV X is

$$\mathrm{Var}(X) = \mathbb{E}\left[\left(X - \mathbb{E}(X)\right)^2\right] = \sum_x \mathbb{p}_X(x) \cdot \left(x - \mathbb{E}(X)\right)^2$$

Theorem. For any $a \in \mathbb{R}$, $\mathrm{Var}(a \cdot X) = a^2 \cdot \mathrm{Var}(X)$

(Proof: Exercise!)

Theorem. $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$

Variance

Theorem. $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$

Proof: $\mathrm{Var}(X) = \mathbb{E}\left[\left(X - \mathbb{E}(X)\right)^2\right]$

$$= \mathbb{E}[X^2 - 2\mathbb{E}(X) \cdot X + \mathbb{E}(X)^2]$$

$$= \mathbb{E}(X^2) - 2\mathbb{E}(X)\mathbb{E}(X) + \mathbb{E}(X)^2$$

$$= \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

(Repeated use of linearity!)

Independent Random Variables

Definition. Two random variables X, Y are **(mutually) independent** if for all x, y,

$$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x) \cdot \mathbb{P}(Y = y)$$

Definition. The random variables $X_1, \dots, X_n$ are **(mutually) independent** if for all $x_1, \dots, x_n$,

$$\mathbb{P}(X_1 = x_1, \dots, X_n = x_n) = \mathbb{P}(X_1 = x_1) \cdots \mathbb{P}(X_n = x_n)$$

Note: No need to check for all subsets, but need to check <u>for all outcomes</u>!

Independence – Variance and Expectation

Theorem. If X, Y independent, $\mathbb{E}(X \cdot Y) = \mathbb{E}(X) \cdot \mathbb{E}(Y)$

Theorem. If X, Y independent, $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

Example – Coin Tosses

We flip n coins, each one heads with probability p

- $X_i = \begin{cases} 1, & i\text{–th outcome is heads} \\ 0, & i\text{–th outcome is tails.} \end{cases}$

$\mathbb{P}(X_i = 1) = p$
$\mathbb{P}(X_i = 0) = 1 - p$

"Bernoulli distributed"

- $Z =$ number of heads

Fact. $Z = \sum_{i=1}^{n} X_i$

Binomial: $\mathbb{P}(Z = k) = \binom{n}{k} p^k (1-p)^{n-k}$

Note: $X_1, \ldots, X_n$ are mutually independent! [Verify it formally!]

$$\Rightarrow \quad \mathrm{Var}(Z) = \sum_{i=1}^{n} \mathrm{Var}(X_i) = n \cdot p(1-p)$$

Independence –Expectation

Theorem. If X, Y independent, $\mathbb{E}(X \cdot Y) = \mathbb{E}(X) \cdot \mathbb{E}(Y)$

$$\mathbb{E}(X \cdot Y) = \sum_{x,y} x \cdot y \cdot \mathbb{P}(X = x, Y = y)$$

$$= \sum_{x,y} x \cdot y \cdot \mathbb{P}(X = x) \cdot \mathbb{P}(Y = y)$$

$$= \left(\sum_{x} x \cdot \mathbb{P}(X = x)\right)\left(\sum_{y} y \cdot \mathbb{P}(Y = y)\right) = \mathbb{E}(X) \cdot \mathbb{E}(Y)$$

Independence – Variance

Theorem. If X, Y independent, $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

$$\mathrm{Var}(X+Y) = \mathbb{E}[(X+Y)^2] - \mathbb{E}(X+Y)^2$$

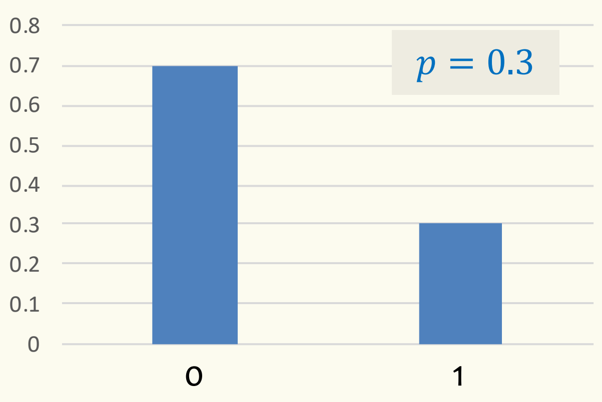
$$= \mathbb{E}[X^2 + 2XY + Y^2] - \big(\mathbb{E}(X) + \mathbb{E}(Y)\big)^2$$

$$= \mathbb{E}(X^2) + 2\mathbb{E}(XY) + \mathbb{E}(Y^2) - \mathbb{E}(X)^2 - 2\mathbb{E}(X)\mathbb{E}(Y) - \mathbb{E}(Y)^2$$

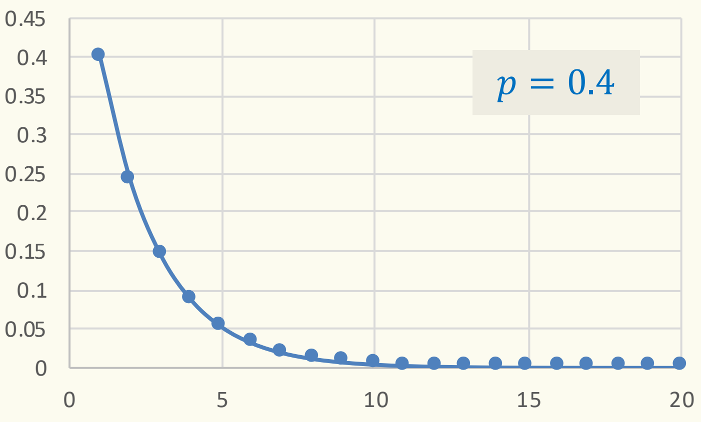
$$= \mathbb{E}(X^2) - \mathbb{E}(X)^2 + \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 \quad = \mathrm{Var}(X) + \mathrm{Var}(Y)$$

Distributions – Recap

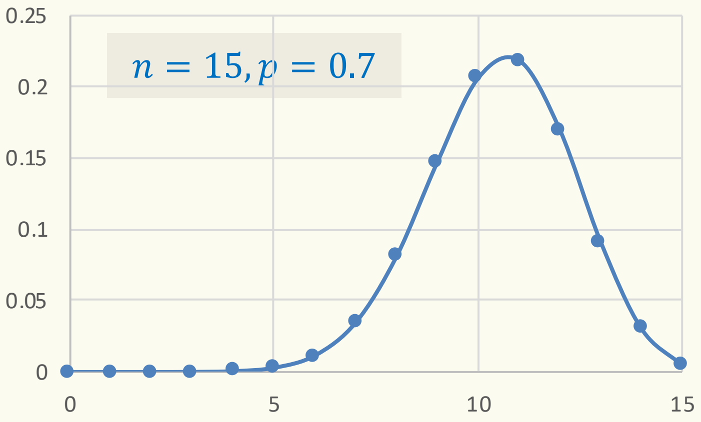
Name	Pars	Range	PMF	Expectation	Variance
Bernoulli	p	$\{0,1\}$	$\mathbb{p}(1) = p,\ \mathbb{p}(0) = 1 - p$	p	$p(1-p)$
Geometric	p	$\{1,2,3,\ldots\} = \mathbb{N}^+$	$\mathbb{p}(i) = (1-p)^{i-1}p$	$1/p$	$(1-p)/p^2$
Binomial	n, p	$\{0,1,\ldots,n\}$	$\mathbb{p}(k) = \binom{n}{k} p^k (1-p)^{n-k}$	np	$np(1-p)$



Bernoulli



Geometric



Binomial