CSE 312

Foundations of Computing II

Lecture 12: Multiple Random Variables, Linearity of Expectation.

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Midterm Information

- Next Friday
  - Closed book, but we will provide important / needed formulas along with the midterm.
  - Covers materials until this Wednesday (HWs 1-4, Sections 0-4).
- Practice midterms online soon
  - They aren't mine, but they will follow a similar spirit.
- Sections will be for midterm review.
- Use edstem for questions.
- Links to extra reading materials / past offerings of CSE312.

Multiple Random Variables

We can define several random variables in the same probability space.

Example: Two dice rolls

- X = 1st outcome
- Y = 2nd outcome
- Z = sum of both outcomes

(joint) probability that $X = x$ **and** $Y = y$

Probability that $X = x$ **conditioned on** $Y = y$

Notation:

- $\mathbb{P}(X = x, Y = y) = \mathbb{P}(\{X = x\} \cap \{Y = y\})$
- $\mathbb{P}(X = x \mid Y = y) = \mathbb{P}(\{X = x\} \mid \{Y = y\})$

Multiple Random Variables

Notation:

- $\mathbb{P}(X = x, Y = y) = \mathbb{P}(\{X = x\} \cap \{Y = y\})$
- $\mathbb{P}(X = x \mid Y = y) = \mathbb{P}(\{X = x\} \mid \{Y = y\})$

Example: Two dice rolls

- X = 1st outcome
- Y = 2nd outcome
- Z = sum of both outcomes

e.g. $\mathbb{P}(X = 3, Z = 6) = \mathbb{P}(\{(3,3)\}) = \frac{1}{36}$, $\mathbb{P}(X = 3 \mid Z = 6) = \frac{1/36}{5/36} = \frac{1}{5}$

Also note: $Z(\omega) = X(\omega) + Y(\omega)$ for all $\omega \in \Omega$

Therefore: We can write $X + Y$ instead of Z

Chain Rule

The chain rule also naturally extends to random variables. E.g.,

$$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x) \cdot \mathbb{P}(Y = y | X = x)$$

Another Example

Example: Two dice rolls

- X_1 = # of times 1 appears
- X_2 = # of times 2 appears

		X_1		
		0	**1**	**2**
X_2	**0**	$4/9$	$2/9$	$1/36$
	1	$2/9$	$1/18$	0
	2	$1/36$	0	0

Joint PMF for X_1 and X_2

$\mathbb{P}(X_1 = a, X_2 = b)$ for all $a, b \in \{0,1,2\}$

Marginal Distribution

The joint PMF of two (or more) RVs gives the PMFs of the individual random variables (aka, the "marginal distribution"). E.g,

$$\mathbb{P}(X = x) = \sum_{y} \mathbb{P}(X = x, Y = y)$$ (Law of total Probability)

X_1

X_2

	0	**1**	**2**
0	4/9	2/9	1/36
1	2/9	1/18	0
2	1/36	0	0

$$\mathbb{P}(X_2 = 0) = \frac{4}{9} + \frac{2}{9} + \frac{1}{36} = \frac{25}{36}$$

$$\mathbb{P}(X_2 = 1) = \frac{2}{9} + \frac{1}{18} = \frac{10}{36}$$

$$\mathbb{P}(X_2 = 2) = \frac{1}{36}$$

Example – Coin Tosses

We flip n coins, each one heads with probability p

- $X_i = \begin{cases} 1, & i\text{–th outcome is heads} \\ 0, & i\text{–th outcome is tails.} \end{cases}$
- $Z =$ number of heads

$$\mathbb{P}(X_i = 1) = p$$
$$\mathbb{P}(X_i = 0) = 1 - p$$

“Bernoulli distributed”

Binomial: $\mathbb{P}(Z = k) = \binom{n}{k} p^k (1-p)^{n-k}$

Fact. $Z = \sum_{i=1}^{n} X_i$

Expectation – Refresher

Definition. The **expectation** of a (discrete) RV X is

$$\mathbb{E}(X) = \sum_x x \cdot p_X(x) = \sum_x x \cdot \mathbb{P}(X = x)$$

Often: $X = X_1 + \cdots + X_n$, and the **RVs** $X_1, \ldots, X_n$ are easier to understand.

Linearity of Expectation

Theorem. For any two random variables X and Y,

$$\mathbb{E}(X+Y)=\mathbb{E}(X)+\mathbb{E}(Y).$$

Or, more generally: For any random variables $X_1, \dots, X_n$,

$$\mathbb{E}(X_1+\cdots+X_n)=\mathbb{E}(X_1)+\cdots+\mathbb{E}(X_n).$$

Because: $\mathbb{E}(X_1+\cdots+X_n)=\mathbb{E}((X_1+\cdots+X_{n-1})+X_n)$

$$=\mathbb{E}(X_1+\cdots+X_{n-1})+\mathbb{E}(X_n)=\cdots$$

Linearity of Expectation – Proof

Theorem. For any two random variables X and Y,
$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.

$$\mathbb{E}(X+Y) = \sum_z z \cdot \mathbb{P}(X+Y=z)$$

$$= \sum_{z_1,z_2} (z_1+z_2) \cdot \mathbb{P}(X=z_1, Y=z_2)$$

$$= \sum_{z_1,z_2} z_1 \cdot \mathbb{P}(X=z_1, Y=z_2) + \sum_{z_1,z_2} z_2 \cdot \mathbb{P}(X=z_1, Y=z_2)$$

$$= \sum_{z_1} z_1 \cdot \sum_{z_2} \mathbb{P}(X=z_1, Y=z_2) + \sum_{z_2} z_2 \cdot \sum_{z_1} \mathbb{P}(X=z_1, Y=z_2)$$

$$= \sum_{z_1} z_1 \cdot \mathbb{P}(X=z_1) + \sum_{z_2} z_2 \cdot \mathbb{P}(Y=z_2)$$

$$= \mathbb{E}(X) + \mathbb{E}(Y)$$

Linearity of Expectation – Even stronger

Theorem. For any random variables $X_1, \ldots, X_n$, and real numbers $a_1, \ldots, a_n \in \mathbb{R}$,

$$\mathbb{E}(a_1 X_1 + \cdots + a_n X_n) = a_1 \mathbb{E}(X_1) + \cdots + a_n \mathbb{E}(X_n).$$

Very important: In general, we do not have $\mathbb{E}(X \cdot Y) = \mathbb{E}(X) \cdot \mathbb{E}(Y)$

Example – Coin Tosses

$\mathbb{E}(X_i) = p \cdot 1 + (1-p) \cdot 0 = p$

We flip n coins, each one heads with probability p

- $X_i = \begin{cases} 1, & i\text{–th outcome is heads} \\ 0, & i\text{–th outcome is tails.} \end{cases}$

$\mathbb{P}(X_i = 1) = p$
$\mathbb{P}(X_i = 0) = 1 - p$

- $Z =$ number of heads

Binomial: $\mathbb{P}(Z = k) = \binom{n}{k} p^k (1-p)^{n-k}$

Fact. $Z = X_1 + \cdots + X_n$

$$\rightarrow \mathbb{E}(Z) = \mathbb{E}(X_1 + \cdots + X_n) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_n) = n \cdot p$$

(Non-trivial) Example – Coupon Collector Problem

Say each round we get a random coupon $X_i \in \{1, \dots, n\}$*, how many rounds (in expectation) until we have one of each coupon?*

Formally: Outcomes in Ω are sequences of integers in $\{1, \dots, n\}$ where each integer appears at least once (+ cannot be shortened).

Example, $n = 3$:

$$\Omega = \{(1,2,3), (1,1,2,3), (1,2,2,3), (1,2,3), (1,1,1,3,3,3,3,3,3,2), \dots\}$$

$$\mathbb{P}\big((1,2,3)\big) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \qquad \mathbb{P}\big((1,1,2,2,2,3)\big) = \left(\frac{1}{3}\right)^6 \qquad \dots$$

Example – Coupon Collector Problem

Say each round we get a random coupon $X_i \in \{1, \ldots, n\}$, how many rounds (in expectation) until we have one of each coupon?

$T_i =$ # of rounds until we have accumulated i distinct coupons

[Aka: length of the sampled ω]

Wanted: $\mathbb{E}(T_n)$

$$Z_i = T_i - T_{i-1}$$

# of rounds needed to go from $i - 1$ to i coupons

Example – Coupon Collector Problem

$T_i =$ # of rounds until we have accumulated i distinct coupons

$Z_i = T_i - T_{i-1}$

$$T_n = T_1 + (T_2 - T_1) + (T_3 - T_2) + \cdots + (T_n - T_{n-1}) = T_1 + Z_2 + \cdots + Z_n$$

$$\Downarrow$$

$$\begin{aligned} \mathbb{E}(T_n) &= \mathbb{E}(T_1) + \mathbb{E}(Z_2) + \mathbb{E}(Z_3) + \cdots + \mathbb{E}(Z_n) \\ &= \underline{1 + \mathbb{E}(Z_2) + \mathbb{E}(Z_3) + \cdots + \mathbb{E}(Z_n)} \end{aligned}$$

Example – Coupon Collector Problem

$T_i =$ # of rounds until we have accumulated i distinct coupons

$Z_i = T_i - T_{i-1}$

If we have accumulated $i-1$ coupons, the number of attempts needed to get the i-th coupon is **geometric** with parameter $p = 1 - \frac{i-1}{n}$.

$$\mathbb{p}_{Z_i}(1) = p \qquad \mathbb{p}_{Z_i}(2) = (1-p)p \qquad \cdots \qquad \mathbb{p}_{Z_i}(i) = (1-p)^{i-1}p$$

➡ $$\mathbb{E}[Z_i] = \frac{1}{p} = \frac{n}{n-i+1}$$

Example – Coupon Collector Problem

$T_i =$ # of rounds until we have accumulated i distinct coupons

$Z_i = T_i - T_{i-1}$ $\qquad$ $\mathbb{E}(Z_i) = \frac{1}{p} = \frac{n}{n-i+1}$

n-th **harmonic number**

$$H_n = \sum_{i=1}^{n} \frac{1}{i}$$

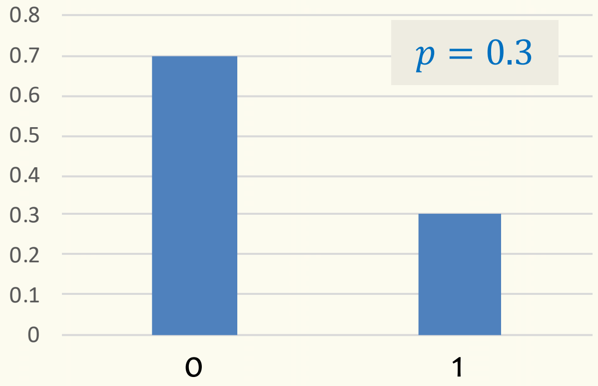
$$\ln(n) \le H_n \le \ln(n) + 1$$

$$\begin{aligned}\mathbb{E}(T_n) &= 1 + \mathbb{E}(Z_2) + \mathbb{E}(Z_3) + \cdots + \mathbb{E}(Z_n)\\ &= 1 + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1}\\ &= n \cdot \left(\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{2} + 1\right) = n \cdot H_n \approx n \cdot \ln(n)\end{aligned}$$

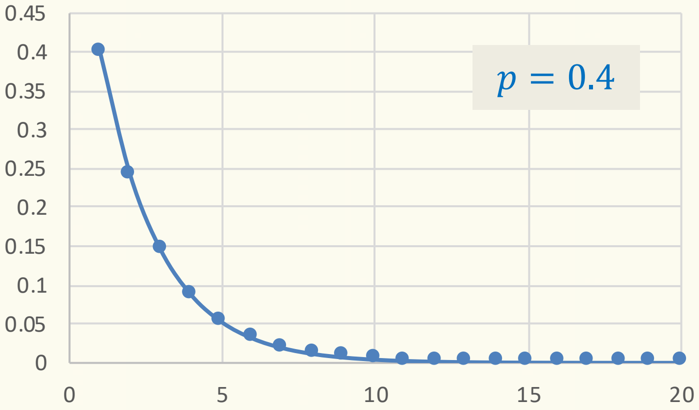
Distributions – Recap

We have encountered some important distributions, let us summarize them

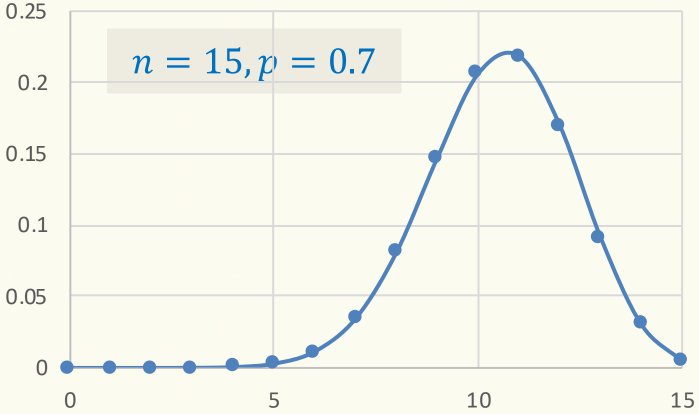
Name	Params	Range	PMF
Bernoulli	p	$\{0,1\}$	$\mathbb{p}(1) = p,\ \mathbb{p}(0) = 1 - p$
Geometric	p	$\{1,2,3,\dots\} = \mathbb{N}^+$	$\mathbb{p}(i) = (1-p)^{i-1}p$
Binomial	n, p	$\{0,1,\dots,n\}$	$\mathbb{p}(k) = \binom{n}{k} p^k (1-p)^{n-k}$



Bernoulli



Geometric



Binomial