CSE 312
Foundations of Computing II

Lecture 12: Multiple Random Variables, Linearity of Expectation.

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Midterm Information

• Next Friday
  – Closed book, but we will provide important / needed formulas along with the midterm.
  – Covers materials until this Wednesday (HWs 1-4, Sections 0-4).

• Practice midterms online soon
  – They aren’t mine, but they will follow a similar spirit.

• Sections will be for midterm review.

• Use edstem for questions.

• Links to extra reading materials / past offerings of CSE312.
Multiple Random Variables

We can define several random variables in the same probability space.

Example: Two dice rolls

- \( X = 1 \text{st outcome} \)
- \( Y = 2 \text{nd outcome} \)
- \( Z = \text{sum of both outcomes} \)

Probability that
\( X = x \)
conditioned on
\( Y = y \)

Notation:
- \( \mathbb{P}(X = x, Y = y) = \mathbb{P}((X = x) \cap (Y = y)) \)
- \( \mathbb{P}(X = x \mid Y = y) = \mathbb{P}((X = x) \mid (Y = y)) \)
Multiple Random Variables

Example: Two dice rolls

- $X = 1st$ outcome
- $Y = 2nd$ outcome
- $Z = sum \ of \ both \ outcomes$

\[ e.g. \ P(X = 3, Z = 6) = P((3,3)) = \frac{1}{36}, \ P(X = 3 \mid Z = 6) = \frac{1/36}{5/36} = \frac{1}{5} \]

Also note: $Z(\omega) = X(\omega) + Y(\omega)$ for all $\omega \in \Omega$

Therefore: We can write $X + Y$ instead of $Z$
The chain rule also naturally extends to random variables. E.g.,

$$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x) \cdot \mathbb{P}(Y = y | X = x)$$
Another Example

Example: Two dice rolls
• \( X_1 \) = # of times 1 appears
• \( X_2 \) = # of times 2 appears

<table>
<thead>
<tr>
<th>( X_2 )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4/9</td>
<td>2/9</td>
<td>1/36</td>
</tr>
<tr>
<td>1</td>
<td>2/9</td>
<td>1/18</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1/36</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Joint PMF for \( X_1 \) and \( X_2 \)
\[ \mathbb{P}(X_1 = a, X_2 = b) \] for all \( a, b \in \{0,1,2\} \)
Marginal Distribution

The joint PMF of two (or more) RVs gives the PMFs of the individual random variables (aka, the “marginal distribution”). E.g,

$$P(X = x) = \sum_y P(X = x, Y = y)$$  \hspace{1cm} \text{(Law of total Probability)}

<table>
<thead>
<tr>
<th>$X_2$</th>
<th>$X_1$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>4/9</td>
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<td>2</td>
<td></td>
<td>1/36</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

- $P(X_2 = 0) = \frac{4}{9} + \frac{2}{9} + \frac{1}{36} = \frac{25}{36}$
- $P(X_2 = 1) = \frac{2}{9} + \frac{1}{18} = \frac{10}{36}$
- $P(X_2 = 2) = \frac{1}{36}$
Example – Coin Tosses

We flip $n$ coins, each one heads with probability $p$

- $X_i = \begin{cases} 1, & i\text{-th outcome is heads} \\ 0, & i\text{-th outcome is tails.} \end{cases}$

- $Z =$ number of heads

Binomial: $\Pr(Z = k) = \binom{n}{k}p^k(1-p)^{n-k}$

Fact. $Z = \sum_{i=1}^{n} X_i$

\[ \Pr(X_i = 1) = p \]
\[ \Pr(X_i = 0) = 1 - p \]

“Bernoulli distributed”
**Definition.** The *expectation* of a (discrete) RV $X$ is

$$
\mathbb{E}(X) = \sum_x x \cdot p_X(x) = \sum_x x \cdot \mathbb{P}(X = x)
$$

Often: $X = X_1 + \cdots + X_n$, and the RVs $X_1, \ldots, X_n$ are easier to understand.
**Linearity of Expectation**

**Theorem.** For any two random variables $X$ and $Y$,

$$\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y).$$

Or, more generally: For any random variables $X_1, \ldots, X_n$,

$$\mathbb{E}(X_1 + \cdots + X_n) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_n).$$

**Because:**

$$\mathbb{E}(X_1 + \cdots + X_n) = \mathbb{E}((X_1 + \cdots + X_{n-1}) + X_n)$$
$$= \mathbb{E}(X_1 + \cdots + X_{n-1}) + \mathbb{E}(X_n) = \cdots$$
**Theorem.** For any two random variables $X$ and $Y$, 
$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

**Linearity of Expectation – Proof**

$$\mathbb{E}(X + Y) = \sum_z z \cdot \mathbb{P}(X + Y = z)$$

$$= \sum_{z_1,z_2} (z_1 + z_2) \cdot \mathbb{P}(X = z_1, Y = z_2)$$

$$= \sum_{z_1,z_2} z_1 \cdot \mathbb{P}(X = z_1, Y = z_2) + \sum_{z_1,z_2} z_2 \cdot \mathbb{P}(X = z_1, Y = z_2)$$

$$= \sum_{z_1} z_1 \cdot \sum_{z_2} \mathbb{P}(X = z_1, Y = z_2) + \sum_{z_2} z_2 \cdot \sum_{z_1} \mathbb{P}(X = z_1, Y = z_2)$$

$$= \sum_{z_1} z_1 \cdot \mathbb{P}(X = z_1) + \sum_{z_2} z_2 \cdot \mathbb{P}(Y = z_2)$$

$$= \mathbb{E}(X) + \mathbb{E}(Y)$$
Theorem. For any random variables $X_1, \ldots, X_n$, and real numbers $a_1, \ldots, a_n \in \mathbb{R}$,

$$\mathbb{E}(a_1X_1 + \cdots + a_nX_n) = a_1\mathbb{E}(X_1) + \cdots + a_n\mathbb{E}(X_n).$$

Very important: In general, we do not have $\mathbb{E}(X \cdot Y) = \mathbb{E}(X) \cdot \mathbb{E}(Y)$.
Example – Coin Tosses

We flip $n$ coins, each one heads with probability $p$

- $X_i = \begin{cases} 1, & \text{i-th outcome is heads} \\ 0, & \text{i-th outcome is tails.} \end{cases}$

- $Z =$ number of heads

**Fact.** $Z = X_1 + \cdots + X_n$

$\mathbb{E}(Z) = \mathbb{E}(X_1 + \cdots + X_n) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_n) = n \cdot p$

$\mathbb{E}(X_i) = p \cdot 1 + (1 - p) \cdot 0 = p$

$\mathbb{P}(X_i = 1) = p$

$\mathbb{P}(X_i = 0) = 1 - p$

**Binomial:** $\mathbb{P}(Z = k) = \binom{n}{k} p^k (1 - p)^{n-k}$
(Non-trivial) Example – Coupon Collector Problem

Say each round we get a random coupon $X_i \in \{1, \ldots, n\}$, how many rounds (in expectation) until we have one of each coupon?

Formally: Outcomes in $\Omega$ are sequences of integers in $\{1, \ldots, n\}$ where each integer appears at least once (+ cannot be shortened).

Example, $n = 3$:

$\Omega = \{(1,2,3), (1,1,2,3), (1,2,2,3), (1,2,3), (1,1,1,3,3,3,3,3,3,3,2), \ldots \}$

$\mathbb{P}((1,2,3)) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}$

$\mathbb{P}((1,1,2,2,2,3)) = \left(\frac{1}{3}\right)^6 \ldots$
Example – Coupon Collector Problem

Say each round we get a random coupon $X_i \in \{1, \ldots, n\}$, how many rounds (in expectation) until we have one of each coupon?

$T_i = \# \text{ of rounds until we have accumulated } i \text{ distinct coupons}$

[Wanted: $E(T_n)$

$Z_i = T_i - T_{i-1}$

# of rounds needed to go from $i - 1$ to $i$ coupons]
Example – Coupon Collector Problem

\( T_i = \# \text{ of rounds until we have accumulated } i \text{ distinct coupons} \)

\( Z_i = T_i - T_{i-1} \)

\[
T_n = T_1 + (T_2 - T_1) + (T_3 - T_2) + \cdots + (T_n - T_{n-1}) = T_1 + Z_2 + \cdots + Z_n
\]

\[
\mathbb{E}(T_n) = \mathbb{E}(T_1) + \mathbb{E}(Z_2) + \mathbb{E}(Z_3) + \cdots + \mathbb{E}(Z_n)
\]

\[
= 1 + \mathbb{E}(Z_2) + \mathbb{E}(Z_3) + \cdots + \mathbb{E}(Z_n)
\]
Example – Coupon Collector Problem

$T_i =$ # of rounds until we have accumulated $i$ distinct coupons

$Z_i = T_i - T_{i-1}$

If we have accumulated $i - 1$ coupons, the number of attempts needed to get the $i$-th coupon is **geometric** with parameter $p = 1 - \frac{i-1}{n}$.

$\mathbb{P}_{Z_i}(1) = p \quad \mathbb{P}_{Z_i}(2) = (1 - p)p \quad \cdots \quad \mathbb{P}_{Z_i}(i) = (1 - p)^{i-1}p$

$\mathbb{E}[Z_i] = \frac{1}{p} = \frac{n}{n - i + 1}$
Example – Coupon Collector Problem

$T_i = \# \text{ of rounds until we have accumulated } i \text{ distinct coupons}$

$Z_i = T_i - T_{i-1}$

$\mathbb{E}(Z_i) = \frac{1}{p} = \frac{n}{n - i + 1}$

$\mathbb{E}(T_n) = 1 + \mathbb{E}(Z_2) + \mathbb{E}(Z_3) + \cdots + \mathbb{E}(Z_n)$

$= 1 + \frac{n}{n - 1} + \frac{n}{n - 2} + \cdots + \frac{n}{1}$

$= n \cdot \left( \frac{1}{n} + \frac{1}{n - 1} + \cdots + \frac{1}{2} + 1 \right) = n \cdot H_n \approx n \cdot \ln(n)$

$n\text{-th harmonic number}$

$H_n = \sum_{i=1}^{n} \frac{1}{i}$

$\ln(n) \leq H_n \leq \ln(n) + 1$
We have encountered some important distributions, let us summarize them.

<table>
<thead>
<tr>
<th>Name</th>
<th>Params</th>
<th>Range</th>
<th>PMF</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bernoulli</td>
<td>$p$</td>
<td>${0,1}$</td>
<td>$p(1) = p, \ p(0) = 1 - p$</td>
</tr>
<tr>
<td>Geometric</td>
<td>$p$</td>
<td>${1,2,3,\ldots} = \mathbb{N}^+$</td>
<td>$p(i) = (1 - p)^{i-1}p$</td>
</tr>
<tr>
<td>Binomial</td>
<td>$n,p$</td>
<td>${0,1,\ldots,n}$</td>
<td>$p(k) = \binom{n}{k}p^k(1-p)^{n-k}$</td>
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