CSE 312

Foundations of Computing II

Lecture 11: Random Variables and their Expectation

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Random Variables

Definition. A **random variable (RV)** for a probability space $(\Omega, \mathbb{P})$ is a function $X: \Omega \to \mathbb{R}$.*

Example. Throwing two dice $\quad \Omega = \{(i, j) \mid i, j \in [6]\} \quad \mathbb{P}\big((i, j)\big) = \frac{1}{36}$.

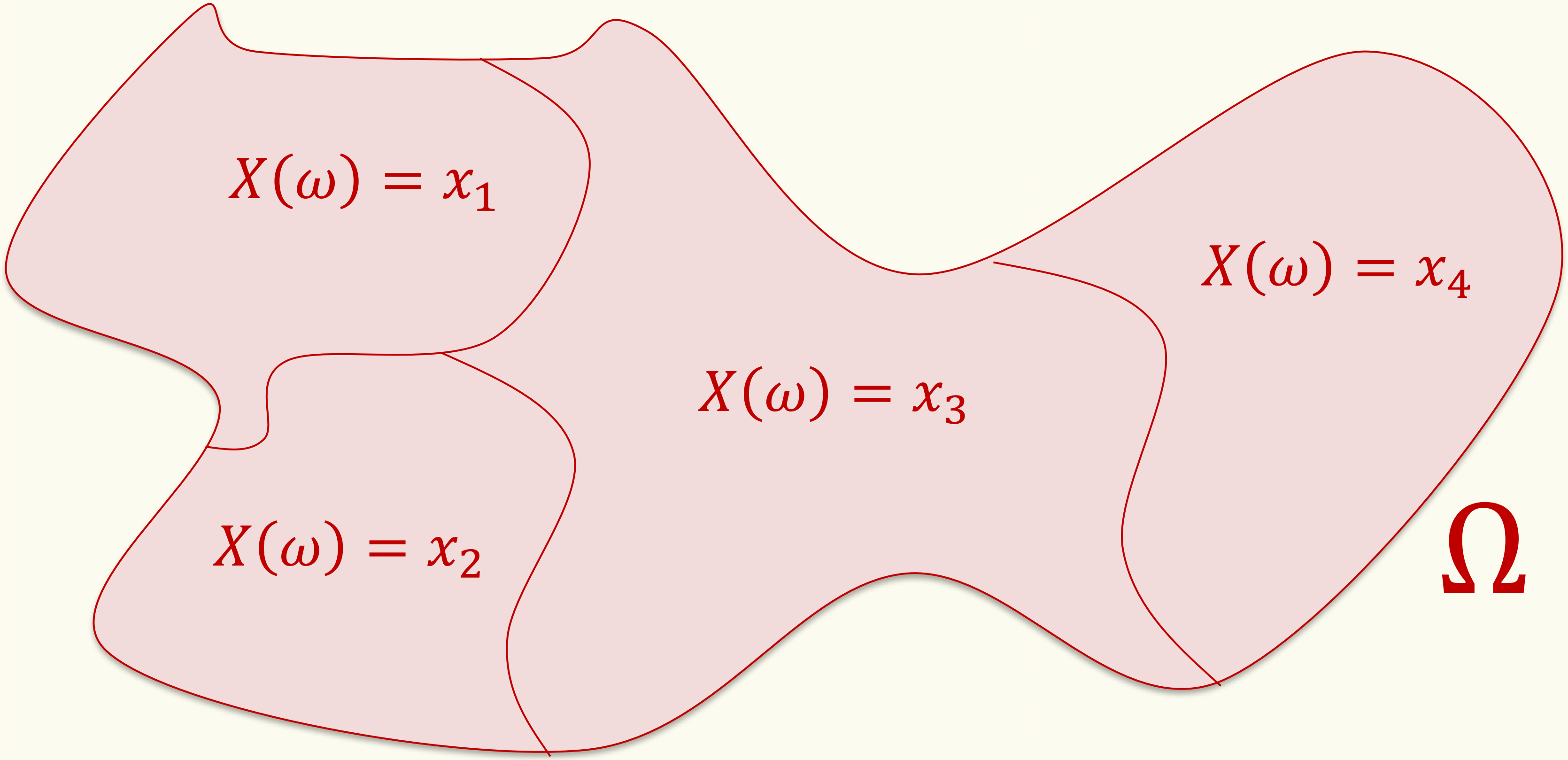
$$\left.\begin{aligned} X(i, j) &= i + j \\ Y(i, j) &= i \cdot j \\ Z(i, j) &= i \end{aligned}\right\}\ \textbf{Random variables!}$$

* random variables outputting values from a non-numeric set can also be defined.

Random Variables and the Probability Space

Random variables partition the sample space.

$X : \Omega \to \mathbb{R}$



Probability Mass Function

Definition. The **range** of a random variable $X: \Omega \to \mathbb{R}$ is

$$X(\Omega) = \{X(\omega) \mid \omega \in \Omega\}$$

i.e., the set of values the random variable can take. If this set is countable, the RV is **discrete**.

Definition. The **probability mass function (PMF)** of a discrete RV $X: \Omega \to \mathbb{R}$ is the function $p_X: X(\Omega) \to \mathbb{R}$ such that for all $x \in X(\Omega)$:

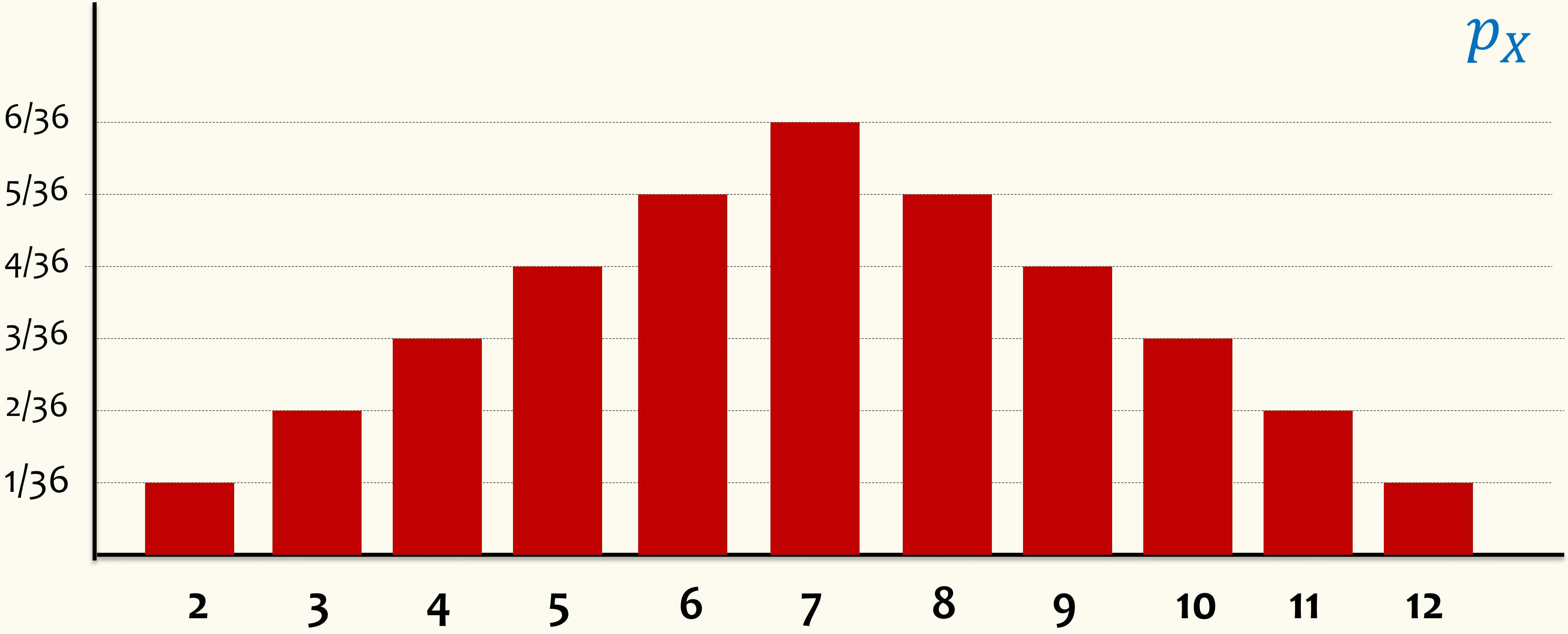
$$p_X(x) = \mathbb{P}(X = x)$$

(often, just the "distribution of X")

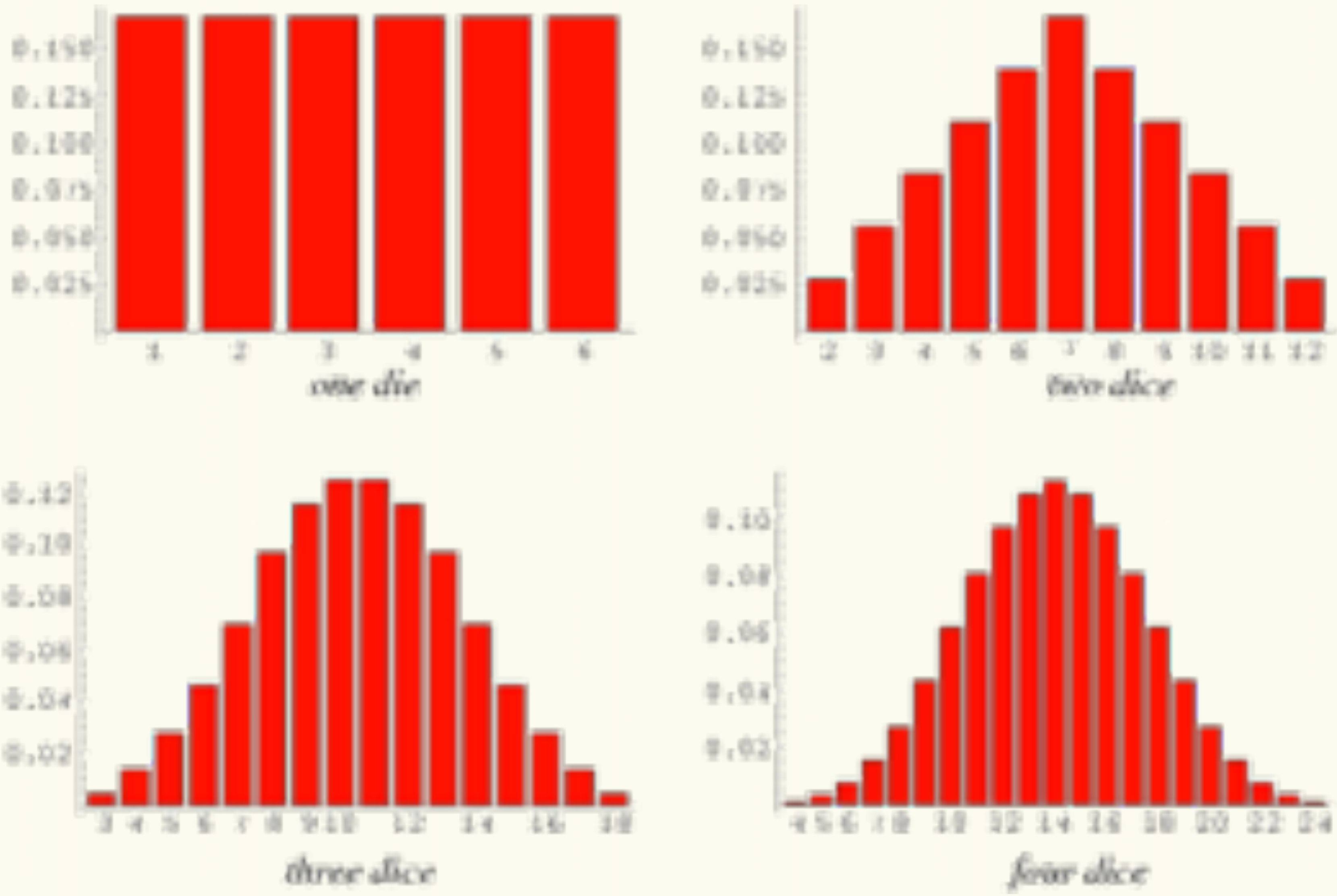
Note: $\sum_x p_X(x) = 1$

Example – Two Dice

X = sum of two dice throws

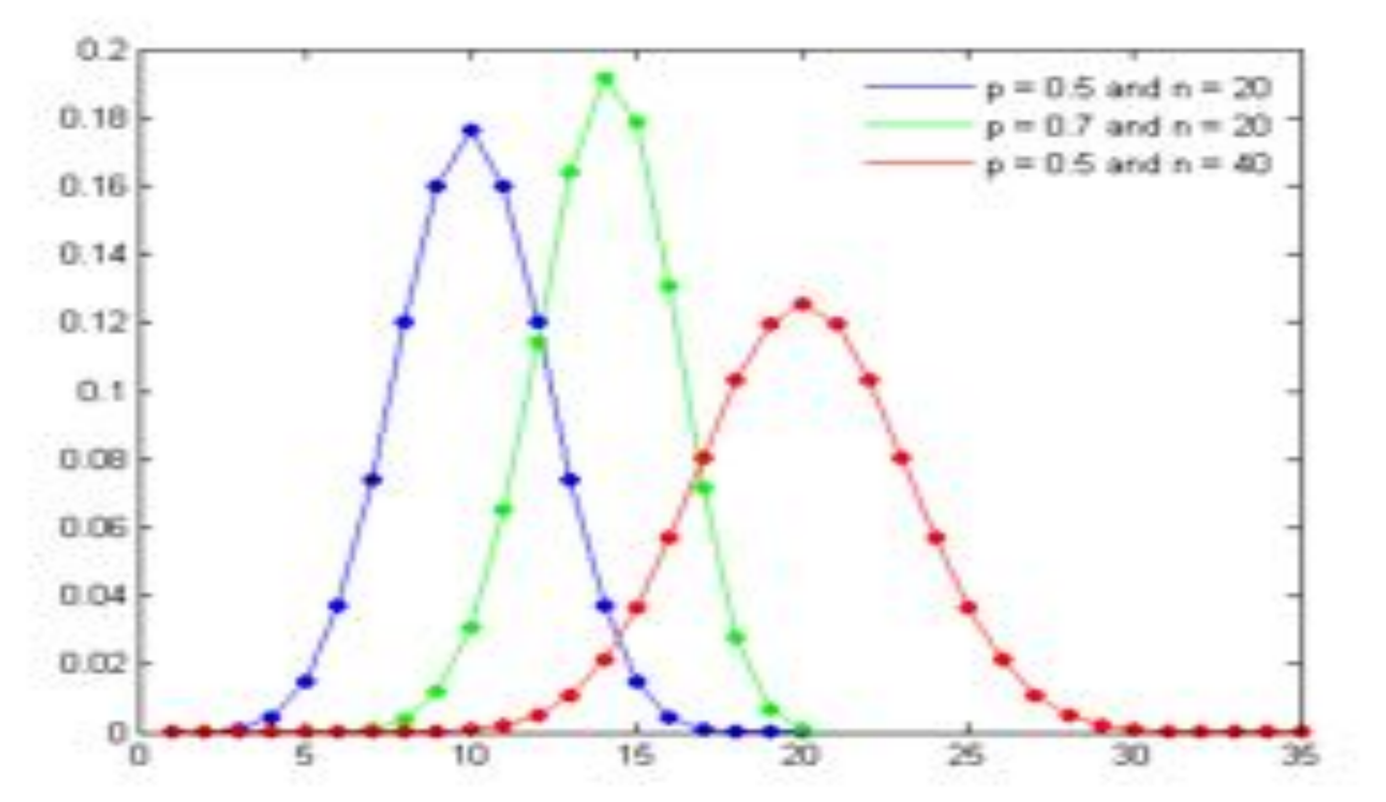


Multiple Dice Throw



Source: wolfram.com

Example – Number of Heads



$$p_X(k) = \mathbb{P}(X = k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$$

Binomial distribution with parameters n and p. Denoted $\mathrm{Bin}(n, p)$

Random Variables as Abstraction

- Often, different probability spaces give random variables with the same distribution.
- We often want to make statements that only depend on the PMF, and hence apply to any of these experiments.

Expectation

Alternatively: **expected value** / **mean** (But NOT “average”)

Definition. The **expectation** of a (discrete) RV X is

$$\mathbb{E}[X] = \sum_x x \cdot p_X(x) = \sum_x x \cdot \mathbb{P}(X = x)$$

Example. Outcome X of rolling one dice

$$p_X(1) = p_X(2) = \cdots = p_X(6) = \frac{1}{6}$$

$$\mathbb{E}[X] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{21}{6} = 3.5$$

For the equally-likely outcomes case, this is just the average of the possible outcomes!

Expectation

Definition. The **expectation** of a (discrete) RV X is

$$\mathbb{E}[X] = \sum_x x \cdot p_X(x) = \sum_x x \cdot \mathbb{P}(X = x)$$

Example. Random variable Z with

$$p_Z(-1) = p_Z(1) = \frac{1}{2}$$

$$\mathbb{E}[Z] = -1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 0$$

Another Interpretation

"If X is how much you win playing the game in one round. How much would you expect to win, <u>on average</u>, per game, when repeatedly playing?"

Answer: $\mathbb{E}[X]$

e.g., two dice rolls, outcome = \$ win – \$3.5 per round win on avg

-1/1 with probability ½ each – \$0 per round win on avg.

Word of warning

Two very different random variables X and Y can have the <u>same</u> expectation $\mathbb{E}[X] = \mathbb{E}[Y]$.

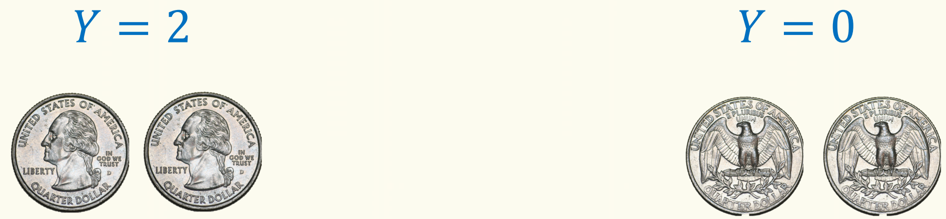
- Expectation is useful, but insufficient to usefully characterize behavior of a random variable.

Example 1. Two independent coin tosses. X = # of heads

$$p_X(0) = \frac{1}{4} \quad p_X(1) = \frac{1}{2} \quad p_X(2) = \frac{1}{4}$$

$$\mathbb{E}[X] = \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 1 + \cdot \frac{1}{4} \cdot 2 = \frac{1}{2} + \frac{1}{2} = \mathbf{1}$$

Example 2. Two completely correlated tosses. Y = # of heads



$$p_Y(0) = p_Y(2) = \frac{1}{2}$$

$$\mathbb{E}[Y] = \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 2 = 0 + 1 = \mathbf{1}$$

Example – Number of trials

$X =$ # of independent coin tosses until we get heads (heads w/ prob p)

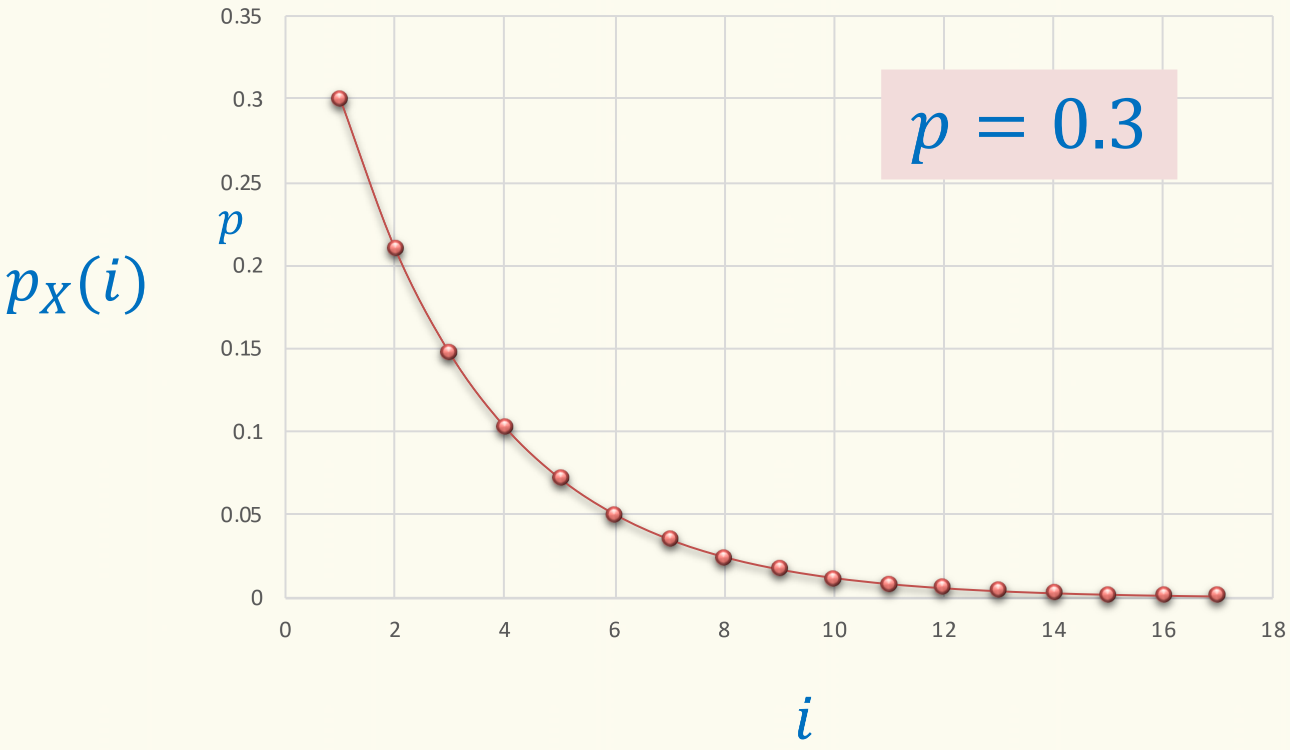
$$\mathbb{P}(X = 1) = p$$

$$\mathbb{P}(X = 2) = (1 - p)p$$

$$\mathbb{P}(X = 3) = (1 - p)^2 p$$

$$\ldots$$

$$\mathbb{P}(X = i) = (1 - p)^{i-1} p$$



“Geometric distribution”

Expectation – Geometric Random Variable

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} p$$

$$\mathbb{P}(X = i) = (1-p)^{i-1} p$$

$$= p \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1}$$

Fact. $\sum_{i=1}^{\infty} i \cdot x^{i-1} = \frac{1}{(1-x)^2}$

$$= p \frac{1}{\left(1-(1-p)\right)^2} = p \frac{1}{p^2} = \frac{1}{p}$$

Expectation – Binomial Random Variable

We flip n coins, independently, each heads with probability p

Y $=$ # of heads

$$\mathbb{P}(Y = k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$$

Expectation – How not to compute it …

$$\mathbb{P}(Y = k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$$

$$\mathbb{E}[Y] = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} k \cdot \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k} \quad = \sum_{k=1}^{n} \frac{n!}{(k-1)!\,(n-k)!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,(n-k)!} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{k=0}^{n-1} \frac{(n-1)!}{k!\,(n-1-k)!} p^k (1-p)^{(n-1)-k}$$

$$= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{(n-1)-k} = np\big(p + (1-p)\big)^{n-1} = np \cdot 1 = np$$

This is only for reference! – we will show next week how to really solve this elegantly.

Multiple Random Variables

We can define <u>several</u> random variables in the same probability space.

Example:

- X = outcome of 1st dice roll
- Y = outcome of 2nd dice roll
- Z = sum of both dice rolls

(joint) probability that
$X = x$ **and** $Y = y$

Notation:

- $\mathbb{P}(X = x, Y = y) = \mathbb{P}(\{X = x\} \cap \{Y = y\})$
- $\mathbb{P}(X = x \mid Y = y) = \mathbb{P}(\{X = x\} \mid \{Y = y\})$

Probability that
$X = x$
conditioned on
$Y = y$

Multiple Random Variables

Notation:

- $\mathbb{P}(X = x, Y = y) = \mathbb{P}(\{X = x\} \cap \{Y = y\})$
- $\mathbb{P}(X = x \mid Y = y) = \mathbb{P}(\{X = x\} \mid \{Y = y\})$

Example:

- X = outcome of 1st dice roll
- Y = outcome of 2nd dice roll
- Z = sum of both dice rolls

e.g. $\mathbb{P}(X = 3, Z = 6) = \mathbb{P}(\{3,3\}) = \frac{1}{36}$

Multiple Random Variables

We roll a dice 2 times. X_i = # of times i appears

Joint PMF for X_1 and X_2

X_1

X_2

	0	1	2
0	$4/9$	$2/9$	$1/36$
1	$2/9$	$1/18$	0
2	$1/36$	0	0

Table entries:

$\mathbb{P}(X_1 = a, X_2 = b)$ for all $a, b \in \{0,1,2\}$