Quiz Section 9

Task 1 – MLE for Gaussians

a) Suppose $x_1, x_2, \ldots, x_n$ are samples from a normal distribution whose mean is known to be zero, but whose variance is unknown. What is the maximum likelihood estimator for its variance?

b) Suppose the mean is known to be μ but the variance is unknown. How does the maximum likelihood estimator for the variance differ from the maximum likelihood estimator when both mean and variance are unknown?

Task 2 – MLE Computation

Let $f(x \mid \theta) = \theta x^{\theta-1}$ for $0 \leqslant x \leqslant 1$, where θ is any positive real number. Let $x_1, x_2, \ldots, x_n$ be i.i.d. samples from this distribution. Derive the maximum likelihood estimator $\hat{\theta}$.

Task 3 – Confidence Interval

Suppose $X_1, \ldots, X_n$ are identically distributed independent random variables with unknown mean θ, and known variance σ^2. Our estimate $\widehat{\theta}$ for the mean θ is the sample mean

$$\widehat{\theta} = \frac{1}{n}\sum_{i=1}^{n} X_i \ .$$

For any α, we want to derive a $100(1-\alpha)\%$ confidence interval (centered around $\widehat{\theta}$) for the true parameter θ, i.e., we are looking for the *smallest* Δ such that

$$\mathbb{P}\left(\widehat{\theta} - \Delta \leqslant \theta \leqslant \widehat{\theta} + \Delta\right) \geqslant 1 - \alpha \ .$$

You may assume n to be sufficiently large for the CLT to apply.

Task 4 – Biased Estimator

Suppose that $\hat{\theta}$ is a biased estimator for θ with $\mathbb{E}\left(\hat{\theta}\right) = \alpha\theta$, for some constant $\alpha > 0$. Find an unbiased estimator for θ and prove that it is unbiased.