Quiz Section 8

Review

1) Gaussian with mean 0 and variance 1: Let X be a random variable distributed according to the Gaussian distribution with mean 0 and variance 1. The PDF is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}.$$

Define $\Phi(x) = p(X \leqslant x) = \int_{-\infty}^{x} f(x)dx$.

2) Central Limit Theorem: Given a real valued distribution with mean μ and variance σ^2, if $X_1, \ldots, X_n$ are drawn independently from this distribution and

$$Y_n = \frac{X_1 + \ldots + X_n - n\mu}{\sigma\sqrt{n}},$$

then the CDF of Y_n converges to the CDF of the Gaussian distribution with mean 0 and variance 1.

3) Moment Generating Function: Let X be a real valued random variable. If M is the moment generating function of X, then $M(t) = \mathbb{E}\left(e^{tX}\right)$.

4) Moment Generating Functions and Distributions: If random variables X, Y have the same moment generating function, then they have the same cumulative distribution function.

Task 1 – Gaussian Distribution

The random variable X is distributed according to the Gaussian distribution with mean 50 and variance 5. Compute $\mathbb{P}(45 \leqslant X \leqslant 52)$ in terms of Φ.

Task 2 – Central Limit Theorem

A factory produces X_i gadgets on day i, where the X_i's are independent and identically distributed random variables, each with mean 5 and variance 9.

a) Using the Central Limit Theorem, approximate the probability that the total number of gadgets produced in 100 days is less than 440. Your solution can be in terms of Φ.

b) Using the Central Limit Theorem, approximate the greatest value of n such that

$$\mathbb{P}(X_1 + X_2 + \ldots + X_n \geqslant 5n + 200) \leqslant 0.05.$$

Your solution can be in terms of Φ^{-1}.

Task 3 – More on the Central Limit Theorem

Suppose $X_1, \ldots, X_n$ are identical and independent random variables distribute according to the Poisson Distribution with parameter λ, and let $X = \frac{\sum_{i=1}^{n} X_i}{n}$, the sample mean. Using the Central Limit Theorem, estimate the smallest value of n such that

$$\mathbb{P}\left(\frac{\lambda}{2} \leqslant X \leqslant \frac{3\lambda}{2}\right) \geqslant 0.99.$$

Task 4 – Moment Generating Functions

a) Explain how to compute the moment $\mathbb{E}\left(X^k\right)$, for any $k \geqslant 1$, from the moment generating function $M(t) = \mathbb{E}\left(e^{tX}\right)$.

b) Compute the moment generating function of the uniform distribution on $[a, b]$.

c) X is known to be a discrete distribution. In addition, we know that the moment generation function of X is given by

$$M(t) = e^{-2t}/4 + e^{-t}/6 + 1/4 + e^{t}/6 + e^{2t}/6.$$

Compute the probability that $|X| \leqslant 1$.