Quiz Section 7

Review

1) Chernoff Bounds. Let $X_1, \ldots, X_n$ be independent and identical random variables in the range between 0 and 1, and let $X = X_1 + \ldots + X_n$ with $\mathbb{E}(X) = \mu$. For any $\varepsilon > 0$,

$$\mathbb{P}(|X - \mu| > \varepsilon\mu) \leqslant 2e^{-\frac{\varepsilon^2\mu}{2+\varepsilon}}.$$

2) Probability Density Function (PDF or density): Let X be a continuous random variable on the Reals. $f_X : \mathbb{R} \to \mathbb{R}$ is defined to be its probability density function that satisfies

- $\forall x$, $f_X(x) \geqslant 0$
- $\int_{-\infty}^{\infty} f_X(x)dx = 1$.

In particular, we have that $\mathbb{P}(a \leqslant X \leqslant b) = \int_a^b f(x)dx$.

3) Cumulative Distribution Function (CDF): Let X be a continuous random variable on the Reals with PDF f_X. $F_X : \mathbb{R} \to \mathbb{R}$ is defined to be its cumulative distribution function that satisfies

$$F_X(x) = \mathbb{P}(X \leqslant x) = \int_{-\infty}^{x} f_X(x)dx.$$

In other words, $f_X(x) = F'_X(x)$.

4) Uniform Random Variable: Let X be a random variable that is uniformly distributed in $[a, b]$. The PDF of X is given by

$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{if } x \in [a, b] \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $\mathbb{E}([X) = (a+b)/2$ and $\mathrm{Var}(X) = (b-a)^2/12$.

5) Exponential Distribution: Let X be a random variable that is distributed according to the Exponential distribution with parameter λ. The PDF of X is given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geqslant 0 \\ 0, & \text{otherwise.} \end{cases}$$

Its CDF is given by

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x}, & \text{if } x \geqslant 0 \\ 0, & \text{otherwise.} \end{cases}$$

6) Memoryless Property: For $s, t \geqslant 0$, if X is distributed according to an Exponential distribution, then $\mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t)$.

Task 1 – Chernoff Bounds

Let a fair coin be tossed n times, in which each toss is independent of the other.

a) What is the probability that the number of heads is less than $n/4$ or more than $3n/4$?

b) What is the probability that the number of heads is at least $n/2 - \sqrt{2n\ln n}$ and at most $n/2 + \sqrt{2n\ln n}$?

c) What does the above answer say about $\sum_{i=n/2-\sqrt{2n\ln n}}^{n/2+\sqrt{2n\ln n}} \binom{n}{i}$?

Task 2 – A New Distribution

Alex decided he wanted to create a *new* type of probability density. Alex wants it to be continuous and have uniform density, and needs help working out some of the details. We will denote a random variable X having the *Uniform-2* distribution as $X \sim \mathsf{Unif}_2(a,b,c,d)$, where $a < b < c < d$. We want the density to be non-zero in $[a,b]$ and $[c,d]$, and zero everywhere else. Anywhere the density is non-zero, it must be equal to the same constant.

a) Find the probability density function, $f_X(x)$. Be sure to specify the values it takes on for every point in $(-\infty,\infty)$.

b) Find the cumulative density function $F_X(x)$. Be sure to specify the values it takes on for every point in $(-\infty,\infty)$.

Task 3 – Dart

You throw a dart at an $s \times s$ square dart board. The goal of this game is to get the dart to land as close to the lower left corner of the dartboard as possible. However, your aim is such that the dart is equally likely to land at any point on the dartboard. Let random variable X be the length of the side of the smallest square B in the lower left corner of the dartboard that contains the point where the dart lands. That is, the lower left corner of B must be the same point as the lower left corner of the dartboard, and the dart lands somewhere along the upper or right edge of B. For X, find the CDF $F_X(x)$, PDF $F_X(x)$, expectation $\mathbb{E}(X)$, and variance $\mathrm{Var}(X)$.

Task 4 – Exponentials

Let $\mathsf{Exp}(\lambda)$ denote the exponential density with the parameter λ, and let $X_1,\ldots,X_n$ be identical and independent random variables, in which X_i follows $\mathsf{Exp}(\lambda_i)$.

a) Show that $X = \min\{X_1,\ldots,X_n\}$ follows $\mathsf{Exp}(\lambda)$ where $\lambda = \lambda_1 + \ldots + \lambda_n$.

b) Suppose I have a device that needs two batteries at all times. We have n batteries, each of which has lifetime following $\mathsf{Exp}(\lambda)$ independently of other batteries. Assume we instantaneously switch to a new battery when one dies. Initially, we use 2 of the n batteries. What is the expected time I can operate this device?

Task 5 – Function of Random Variables

Let X be the uniform distribution in $[0,1]$. If $Y = e^X$, then find the PDF of Y, $f_Y(y)$