Quiz Section 6

Review

1) Variance. $\mathrm{Var}(aX) =$ ________.

2) Conditional Expectation. For an event $\mathcal{A}$ and a random variable X, defined over the same probability space, we have

$$\mathbb{E}(X \mid \mathcal{A}) = \underline{\hspace{6cm}}\ .$$

3) Law of total expectation. Let X be a random variable, and $\mathcal{A}_1, \ldots, \mathcal{A}_n$ a partition of the underlying sample space. Then,

$$\mathbb{E}(X) = \underline{\hspace{6cm}}\ .$$

4) Poisson Distribution. If X is a Poisson random variable with parameter λ, then

$$\mathbb{P}(X = i) = \underline{\hspace{4cm}}\ .$$

5) Markov's inequality. If X is a non-negative random variable, then $\mathbb{P}(X \geqslant t) \leqslant$ ________.

6) Chebyshev's inequality. If X is a random variable with expectation μ and variance σ^2, then

$$\mathbb{P}(|X - \mu| \geqslant t) \leqslant \underline{\hspace{2cm}}\ .$$

Task 1 – Hard Expectations

Let N be a random variable which can only take non-negative integers as values, and which has mean $\mathbb{E}(N) = \alpha$. Let $X_1, X_2, \ldots$ be independent and identically distributed (i.i.d.) random variables with mean μ and variance σ^2. Also, N is independent of $X_1, X_2, \ldots$. Let $X = \sum_{i=1}^{N} X_i$.

a) Compute $\mathbb{E}(X)$.

b) Compute $\mathrm{Var}(X)$ assuming $\mu = 0$.

Task 2 – Poisson

a) The ER at a major hospital admits an average of 12 patients every 3 hours. What is the probability that at least 2 patients are admitted to ER within a particular hour, assuming that this number follows the Poisson model?

b) Let X_1 and X_2 be independent Poisson random variables with parameters λ_1 and λ_2, respectively. Show that $X = X_1 + X_2$ is a Poisson random variable with parameter $\lambda_1 + \lambda_2$.

Hint: $\mathbb{P}(X = k) = \sum_{i=0}^{k} \mathbb{P}(X_1 = i) \cdot \mathbb{P}(X_2 = k - i)$, and the binomial theorem.

Task 3 – Relaxed Markov

Let X be a random variable such that $\mathbb{E}(X) > 0$ (but X itself can take negative values). Is it true that $\mathbb{P}(X \geqslant t) \leqslant \mathbb{E}(X)/t$?

Task 4 – Concentration

Within each hour, 400 cars are expected to drive through the intersection of N 45th St and 15th Ave N.

a) Give an upper bound on the probability that more than 800 cars drive through the intersection, assuming you have no further information about the distribution of the number of cars driving through the intersection.

b) Can you improve the above upper bound if you assume that the number of cars driving through the intersection follows a Poisson Distribution?

Task 5 – The Sample Mean

Let $X_1, \ldots, X_n$ be independent and identically distributed (i.i.d.) random variables with mean μ and variance σ^2. Define $X = \frac{1}{n}\sum_{i=1}^{n} X_i$; this is usually referred to as the *sample mean*.

a) Compute $\mathbb{E}(X)$ and $\operatorname{Var}(X)$, as a function of n, μ and σ^2.

b) How large does n need to be so that $\mathbb{P}(|X - \mu| \geqslant \sigma) \leqslant 0.01$?

Task 6 – The Hypergeometric Distribution

Consider an urn with N balls, $K \leqslant N$ of which are red, and the remaining ones are blue. Assume we draw n balls without replacement. Let X denote the number of red balls among these n balls.

a) Give an exact expression for $\mathbb{P}(X = k)$ for any $k = \max\{0, n + K - N\}, \ldots, \min\{K, n\}$.

Note: A random variable with this PMF is known as **hypergeometric** with parameters N, K, n.

b) Show that $\mathbb{E}(X) = n \cdot K/N$.

Hint. Write $X = X_1 + \cdots + X_n$, where $X_1, \ldots, X_n$ are *non-independent* Bernoulli random variables.