Quiz Section 4

Review

1) Probability Mass. For every random variable X, we have $\sum_x \mathbb{P}(X = x) =$ ______.

2) Expectation. $\mathbb{E}(X) =$ ______.

3) Linearity of expectation. For any random variables $X_1, \ldots, X_n$, and real numbers $a_1, \ldots, a_n$,

$$\mathbb{E}(a_1 X_1 + \cdots + a_n X_n) = \underline{\hspace{6cm}} .$$

4) Variance. $\mathrm{Var}(X) =$ ______.

5) Independence. Two random variables X and Y are **independent** if ____________.

6) Variance and Independence. For any two independent random variables X and Y,

$$\mathrm{Var}(X + Y) = \underline{\hspace{6cm}} .$$

Task 1 – Random Variables

Assume that we roll a fair 3-sided die three times. Here, the sides have values $1, 2, 3$.

a) Describe the PMF of the random variable X giving the sum of the first two rolls.

b) Give the expectation $\mathbb{E}(X)$.

c) Compute $\mathbb{P}(X > 3)$.

d) Let Y be the random variable describing the sum of the three rolls. Describe the joint PMF of X and Y.

e) Compute $\mathbb{P}(X = 5 \mid Y = 7)$.

Task 2 – Servers

A web service uses m identical servers for load balancing. Every web request is assigned to one of the servers independently and uniformly at random. Assume the web service receives n requests.

a) For any $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, define the Bernoulli random variable $X_{i,j}$ which is one if and only the j-th request is assigned to server i, and zero otherwise.

What is the expected values $\mathbb{E}(X_{i,j})$?

b) What is the expected load of server $i \in \{1, \ldots, m\}$?

Task 3 – More Linearity

a) Alice rolls a fair, six-sided die n times, what is the expectation of the sum of the n outcomes?

b) Bob plays a game where a die is rolled in each round, until 6 comes out. He wins \$3 every time 6 does not appear. How much does Bob expect to win?

c) In a room with n people, how many groups of three people are expected to have the same birthday? (Assuming birthdays are independent, and equally liked for each of the n people. Further, assume there are only 365 days.)

Task 4 – Expectations, Independence, and Variance

a) Give random variables X and Y (via their joint PMF) such that $\mathbb{E}(X \cdot Y) \neq \mathbb{E}(X) \cdot \mathbb{E}(Y)$.

b) Give a random variable X with range $\{-1, 1\}$ such that $\mathbb{E}(X)^2 \neq \mathbb{E}(X^2)$.

c) Let U be a random variable which is uniform over the set $[n] = \{1, 2, \ldots, n\}$, i.e, $\mathbb{P}(U = i) = \frac{1}{n}$ for all $i \in [n]$. Compute $\mathbb{E}(U^2)$ and $\mathrm{Var}(U)$.

Hint: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

d) Let Y_1 and Y_2 be the independent outcomes of two dice rolls, and let $Z = Y_1 + Y_2$. Then, compute $\mathbb{E}(Z^2)$ and $\mathrm{Var}(Z)$.

Hint: Try to use an indirect solution using linearity and independence, without the need of explicitly giving the distribution of Z^2.